PHD

# Catalytic superprocesses, collision local times and non-linear boundary value problems

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Submitted by

**Pascal Vogt**

for the degree of Doctor of Philosophy
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*"Es macht nicht allzuviel Schwierigkeit, sich zu beruhigen bei dem, was man schon weiß (...); wer sich aber dem Ganzen Wahrheit bis ins Innerste öffnet, der erwartet, da er von nichts das Ganze sieht, über das bereits Gewußte hinaus, immer noch ein neues Licht."*

*Josef Pieper, Werke IV*

*To my parents*
*and Martina*

## Summary

This thesis presents an independent construction of catalytic superprocesses. Based on a time change technique for generic particles, a random measure is constructed and shown to be equal in law with the collision local time between the catalytic process and its catalyst. The catalytic superprocess is then constructed deterministically from this random measure. This technique also applies to give a direct construction of the exit measure associated to a catalytic super-Brownian motion in a bounded domain with reflecting boundary. The dual function of the exit measure is shown to solve Laplace's equation with a set of mixed non-linear boundary conditions.

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# Contents

# Chapter 1

# Introduction

Measure valued branching processes, such as branching Brownian motion or super-Brownian motion have been a fruitful area of research throughout the last thirty years. They arise — as building blocks for models — both in population genetics and statistical mechanics, and in particular they are studied on their own right from a purely mathematical point of view. See e.g. the recent surveys [Dy94, LG99, Et00].

Informally, combining spatial motion with branching, generic particles move independently in space according to a given spatial motion and split or die according to a given branching mechanism. Roughly speaking, branching Brownian motion is a combination of Brownian motion which describes the spatial motion and a continuous time Galton-Watson process which governs the branching. Describing super-Brownian motion, which can be seen as the continuum limit of branching Brownian motion, the Galton-Watson process has to be replaced by a continuous state branching process.

In recent years, much attention has been given to more complicated models often featuring an intrinsic singular behaviour arising from additional interactions. An important example of additional interaction is *catalytic branching.* Here the branching is governed by a measure $\sigma$ and generic particles may only split or die when they hit the support of this measure. These processes were introduced in [DF91, DF92] and [Dy91], and have attracted considerable interest since; for a recent survey see [Kl99].

This thesis deals with a class of catalytic superprocesses where the catalyst $\sigma$ is a deterministic and time-independent measure on $\mathbb{R}^d$.

The close relation between superprocesses and a class of semi-linear partial differential equations is an interesting and widely studied feature (see e.g. [De97, LG99, Dy02]). In our setting, the log-Laplace transform

$$u(t,x) := -\log \mathbb{E}\big[\exp -\langle X_t, \varphi\rangle \,\big|\, X_0 = \delta_x\big]$$

of a catalytic super-Brownian motion $X = (X_t, t \geq 0)$ with catalyst $\sigma$ is the unique *non-negative* solution of the informal parabolic equation

$$\begin{cases} \partial_t u(t,x) \;=\; \frac{1}{2}\Delta u(t,x) - \sigma(dx)\, u^2(t,x) & \text{in } [0,\infty) \times \mathbb{R}^d \\ \qquad\qquad u(0,\cdot) = \varphi, & \end{cases} \tag{1.1}$$

where $\varphi$ is a bounded non-negative function defined on $\mathbb{R}^d$. Roughly speaking, the Laplacian generates the spatial Brownian motion of generic particles and the informal term $\sigma(dx)$ is the branching rate in a quadratic branching mechanism $\psi(u) = u^2$. Hence, branching is only possible on the support of the catalyst $\sigma$. Of course, to be rigorous, we have to write (1.1) in its mild form as an integral equation (see Theorem 1.3). Nevertheless, equation (1.1) is an illustrative way saying that super-Brownian motion combines a spatial motion with branching.

In this thesis we propose — under certain regularity assumptions — a construction of $\sigma$-catalytic super-Brownian motion (Chapter 3), which is based on a representation of the collision local time of the catalytic super-Brownian motion with its catalyst (Chapter 2). Our results generalize those of [FLG95] and are related to a representation theorem of catalytic super-Brownian motion obtained by [De96].

Much attention has also been given to the Dirichlet problem of the stationary elliptic equation associated to (1.1). Especially the case of *non*-catalytic super-Brownian motion, where the branching term $\sigma u^2$ on the right hand side of (1.1) is replaced by $u^2$, has been studied intensively (see e.g. [Dy91], [LG99]). Here, various techniques — among them DYNKIN'S *exit measures* and LE GALL'S *Brownian snake* — have been developed to give a probabilistic representation of non-negative solutions to the non-linear Dirichlet boundary value problem of the form

$$\begin{cases} \Delta u = 2u^2 & \text{in } D \\ \;\; u = \varphi & \text{on } \partial D \end{cases},$$

where $D$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial D$ and $\varphi$ is a continuous function on $\partial D$.

The techniques developed in Chapter 3 to construct catalytic super-Brownian motion are then used in Chapter 4 to give a probabilistic representation of (weak) solutions to the mixed Dirichlet non-linear Neumann boundary value problem (DNP)

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \varphi & \text{on } F_2 \\ \partial_n u + 2u^2 = 0 & \text{on } F_1 \end{cases},$$

where $D$ is a bounded domain in $\mathbb{R}^d$, $(F_1, F_2)$ is a non-trivial partition of the boundary $\partial D$ of $D$ and $\partial_n$ denotes the outward normal derivative on the boundary. Moreover, we show that the Dirichlet condition on $F_2$ can also be replaced by a Neumann condition.

We start reviewing catalytic super-Brownian motion in Section 1.1. Section 1.2 gives a more detailed overview on our main results.

## 1.1 Spatial branching processes governed by catalysts

In Section 1.1.1 we remind the reader of the construction of catalytic super-Brownian motion as the high density limit of a catalytic branching particle system and intend to introduce and fix some notation. In Section 1.1.2, we define the collision local time $\mathcal{L}_{\sigma,X}$ of a $\sigma$-catalytic super-Brownian motion and its catalytic measure $\sigma$, as it was introduced by JEAN-FRANÇOIS DELMAS in [De96]. It turns out that $\mathcal{L}_{\sigma,X}$ is the fundamental object to study in order to understand catalytic super-Brownian motion (also see e.g. the recent works [EP98, FD01]). Roughly speaking, $\mathcal{L}_{\sigma,X}$ is a random measure, which keeps track of the amount of collisions of infinitesimal particles with the catalytic measure $\sigma$.

### 1.1.1 Catalytic branching particle systems and superprocesses

To define catalytic branching Brownian motion or super-Brownian motion, the right mathematical object to model the *catalyst* is a measure $\sigma$ on $\mathbb{R}^d$. Of course, not all measures are suitable as catalysts, e.g. generic particles which move according to a $d$-dimensional Brownian motion $W = (W_t, t \geq 0)$ should hit the support of $\sigma$. Hence, we have to impose some condition on $\sigma$. In this thesis, we restrict ourselves to the following class of catalysts which has been set up in [De96]: Let $\sigma$ be a non-vanishing sigma-finite measure on the Borel sets of the Euclidian space $(\mathbb{R}^d, |\cdot|)$ such that there is a $\beta \in (0,1)$ satisfying $d-2+2\beta \geq 0$ and

$$\sup_{x\in\mathbb{R}^d} \int_{B_1(x)} \frac{\sigma(dy)}{|x-y|^{d-2+2\beta}} < \infty, \tag{1.2}$$

where $B_1(x)$ denotes the open ball around $x$ with radius 1. In particular, in dimension $d=1$ condition (1.2) is fulfilled whenever $\sigma$ is a *finite* measure (just take $\beta = 1/2$). Moreover, (1.2) implies that $\sigma$ does *not* charge polar sets of $W$ (see e.g. [PS78, Chapter 6]) — as already mentioned, a natural condition on catalytic measures. Roughly speaking, by (1.2) the support of $\sigma$ is not too small. Indeed, (1.2) implies that it has at least Hausdorff dimension $d-2+2\beta$ (see e.g. [Fa90, Chapter IV]).

Recall that a *continuous additive functional* $A = (A_t, t \geq 0)$ of the Brownian motion $W$ is a 1-dimensional stochastic process which is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ generated by $W$ (and completed the usual way) with continuous increasing paths and the additivity property,

$$A_{t+s} = A_t + A_s \circ \theta_t, \tag{1.3}$$

where $\theta_t$ is the family of shift operators $\theta_t\omega_r = \omega_{t+r}$. See Section 2.1.1 for a more precise definition and some properties of continuous additive functionals. Moreover, denote by $\mathbb{P}_x$ the law of the Brownian motion $W$ starting at $x \in \mathbb{R}^d$ and by $\mathbb{E}_x$ the corresponding expectation.

Here, condition (1.2) implies that the measure $\sigma$ has bounded and continuous $\alpha$-potentials for any $\alpha > 0$ (see Section 2.1.2 for the details), i.e. for all $x \in \mathbb{R}^d$ we have

$$u_\sigma^\alpha(x) := \int \sigma(dy) \int_0^\infty ds\ \mathrm{e}^{-\alpha s}\, p_s(x,y) < \infty,$$

where

$$p_s(x,y) := \frac{1}{(2\pi s)^{d/2}} \exp -\frac{|y-x|^2}{2s}, \qquad \text{for } s \in (0,\infty),\ x,y \in \mathbb{R}^d, \tag{1.4}$$

denotes the $d$-dimensional Brownian transition density. As $u_\sigma^\alpha$ is a regular $\alpha$-potential (see Section 2.1.2) the Volkonskii-Sũr-Meyer Theorem ensures the existence of a *unique*, continuous additive functional $A = (A_t, t \geq 0)$ of the Brownian motion $W$ (independent of $\alpha$) such that for all $x \in \mathbb{R}^d$,

$$\mathbb{E}_x\Big[\int_0^\infty dA_s\ \mathrm{e}^{-\alpha s}\Big] = \int \sigma(dy) \int_0^\infty ds\ \mathrm{e}^{-\alpha s}\, p_s(x,y).$$

Hence, [BlG68, Theorem VI.3.1] and a monotone class argument imply that for all non-negative, measurable and bounded functions $\varphi \in \mathcal{B}_+^b(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\mathbb{E}_x \left[ \int_0^\infty dA_s \, \varphi(s, W_s) \right] = \int_0^\infty ds \int \sigma(dy) \, p_s(x,y) \varphi(s,y). \tag{1.5}$$

If (1.5) holds for an additive functional $A$ and a measure $\sigma$, we call $\sigma$ the *Revuz-measure* associated to the continuous additive functional $A$. Hence, we can describe the catalyst *either* by the measure $\sigma$ *or* the associated continuous additive functional $A$.

**Definition 1.1 (The class of uniformly non-polar measures).**
*A measure $\sigma$ on $\mathbb{R}^d$ which satisfies (1.2) is called* uniformly non-polar *in this thesis. Uniformly non-polar measures $\sigma$ are suitable as catalysts. If $\sigma$ is a catalyst, then the associated additive functional $A$ is said to be the* catalytic *or the* branching functional.

Let us give some examples of catalytic measures and the corresponding branching functionals:

(i) If $d = 1$, every *finite* measure $\sigma \in \mathcal{M}_f(\mathbb{R})$ is uniformly non-polar. Moreover, if we denote by $(\ell_t^x, x \in \mathbb{R}, t \geq 0)$ the *local time* of the Brownian motion $W$ then the continuous additive functional $A$ having Revuz-measure $\sigma$ is given explicitly (see e.g. [RY99, Chapter X.2]) by,

$$A_t = \int_{\mathbb{R}} \sigma(dx) \, \ell_t^x.$$

(ii) For any dimension $d \geq 1$, the $d$-dimensional Lebesgue measure $\lambda^d$ is uniformly non-polar.

(iii) Let $d \geq 2$, and $\mathcal{S}$ be a smooth and compact $(d-1)$-dimensional manifold in $\mathbb{R}^d$. If $\sigma$ denotes the surface measure on $\mathcal{S}$ one can check that $\sigma$ is uniformly non-polar. Moreover, the continuous additive functional $A$ associated with $\sigma$ is the *local time* on $\mathcal{S}$ which can be defined by

$$A_t = \lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_0^t ds \, \mathbf{1}_{\{x \in \mathbb{R}^d \, : \, d(x, \mathcal{S}) \leq \varepsilon_n\}}(W_s), \tag{1.6}$$

where $d(x, \mathcal{S}) := \inf\{|x - y| \, : \, y \in \mathcal{S}\}$ denotes the distance of $x$ and $\mathcal{S}$, and the limit exists for all $t \geq 0$, $\mathbb{P}_x$-almost surely for a positive sequence $(\varepsilon_n, n \geq 1)$ decreasing to 0 which does not depend on $x$ (see Theorem 7.2 in [ST62]).

Intuitively, it is often useful to think of catalytic super-Brownian motion as the high density limit of a catalytic branching particle system. Let $\sigma$ be a catalytic measure on $\mathbb{R}^d$ and $A$ be the associated branching functional. We consider a particle system governed by the following rules:

- at time $t = 0$ the particles are distributed according to a (possibly random) finite point measure $\eta \in \mathcal{M}_f^p(\mathbb{R}^d)$;
- given its birth time $t$ and place $x$, each particle moves according to the law of a Brownian motion started at $x$ at time $t$. Moreover, all particles move independently;
- given the trajectory of a particle up to time $s$ and that the particle is alive at time $s$, the probability of survival on the interval $(s, t)$ is given by $e^{-(A_t - A_s)}$;
- a particle which dies gives rise to 0 or 2 children each with probability 1/2 (critical branching) born at the place and time where their mother died.

If we denote by $Y_t(C)$ the number of particles situated in the Borel set $C \in \mathcal{B}(\mathbb{R}^d)$ at time $t$, then $Y = (Y_t, t \geq 0)$ defines a time homogenous Markov process with values in finite point measures $\mathcal{M}_f^p(\mathbb{R}^d)$ on $\mathbb{R}^d$.

**Definition 1.2 (Catalytic branching Brownian motion).**
*The measure valued Markov process $Y$ is called $\sigma$*-catalytic branching Brownian motion.

It is easy to show that $Y$ is a *measure valued branching process*, i.e. if $Y^1$ and $Y^2$ are two independent copies of $Y$ starting from $\eta_1$ and $\eta_2$ respectively, then $Y^1 + Y^2$ has the same law as $Y$ starting from $\eta_1 + \eta_2$.

The total mass process $|Y| = (Y_t(\mathbb{R}^d), t \geq 0)$ of a $\sigma$-catalytic super-Brownian motion $Y$ is a stochastic process with values in $\mathbb{N}$ but in general *not* a Markov or Galton-Watson branching process, as the branching activity is not homogenous in space.

Notice that as the branching functional $A$ only increases whenever the Brownian motion $W$ hits the (fine) support $\mathcal{S}$ of the catalytic measure $\sigma$ (see Section 2.1.2 for a precise statement), branching occurs *only* on the set $\mathcal{S}$ of the catalyst. If we choose $\sigma$ to be Lebesgue measure on $\mathbb{R}^d$, then the associated branching functional $A$ becomes the identity $A_t \equiv t$ and the $\sigma$-catalytic branching Brownian motion degenerates to the usual, *non-catalytic* binary branching Brownian motion.

One may guess that the catalytic branching particle system possesses a superprocess limit as in the non-catalytic case. In that case, both increasing the number of particles together with their branching rate and keeping the expected total mass constant yields the well known particle picture approximation of superprocesses. Let us give a rigorous statement for the catalytic case: Denote by $\eta_n \subseteq \mathcal{M}_f^p(\mathbb{R}^d)$ a sequence of finite point measures and let $Y^n = (Y_t^n, t \geq 0)$ be the sequence of catalytic branching Brownian motions with branching functionals $A^n := nA$ and initial values $Y_0^n := \eta_n$ respectively.

For a fixed catalytic measure $\sigma$, we need the family of constants $(a_T, T \geq 0)$ given by

$$a_T := \sup_{x \in \mathbb{R}^d} \mathbb{E}_x[A_T].$$

Notice, as $\sigma$ is uniformly non-polar, we have $a_T < \infty$ for all $T > 0$ (see Lemma 2.18). Let $\mathcal{M}_f(\mathbb{R}^d)$ be endowed with the topology of weak convergence. We have the following convergence theorem. A proof can be found e.g. in [De96].

**Theorem 1.3 (Convergence of catalytic branching Brownian motion).**
*Assume that $\frac{1}{n}\eta_n$ converges weakly to $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. Then the finite dimensional marginal distributions of the sequence $(\frac{1}{n}Y^n)_{n\in\mathbb{N}}$ converge to those of a process $X = (X_t, t \geq 0)$. Moreover, $X$ is a time homogenous Markov process taking values in $\mathcal{M}_f(\mathbb{R}^d)$ whose law $\mathbb{P}_\eta^X$ can be characterized by $X_0 = \eta$ almost surely and for all $T > 0$, all $0 = t_0 < t_1 < \cdots < t_k < T$ and all $\varphi \in \mathcal{B}_b^+(\mathbb{R}^d)$ such that $2a_T\|\varphi\|_\infty < 1$ we have*

$$\mathbb{E}_\eta^X\left[\exp -\langle X_{t_k}, \varphi\rangle \,\middle|\, X_{t_0}, \ldots, X_{t_{k-1}}\right] = \exp -\big\langle X_{t_{k-1}}, v(t_k - t_{k-1}, \cdot)\big\rangle,$$

*where $v$ is the unique non-negative solution of the integral equation*

$$v(t,x) + \mathbb{E}_x\left[\int_0^t dA_s\, v^2(t-s, W_s)\right] = \mathbb{E}_x[\varphi(W_t)], \tag{1.7}$$

*where $(t,x) \in [0,T] \times \mathbb{R}^d$.*

Moreover, by [De96, Théorème II.4.7], under $\mathbb{P}_\eta^X$ there exists a continuous version of $X$ and we will always work with this version. Let us denote by $(\mathcal{G}_t, t \geq 0)$ the $\mathbb{P}_\eta^X$-completed filtration generated by $X$. Thanks to Theorem 1.3 we pass to the following definition:

**Definition 1.4 ($\sigma$-catalytic super-Brownian motion).**
*The continuous version of the measure valued Markov process $X$ obtained by Theorem 1.3 is called* $\sigma$-catalytic super-Brownian motion. *Indeed, for $0 \leq s \leq t \leq T$ and*

*all* $\varphi \in \mathcal{B}_b^+(\mathbb{R}^d)$ *such that* $2a_T\|\varphi\|_\infty < 1$,

$$\mathbb{E}_\eta^X\big[\exp -\langle X_t, \varphi\rangle \,\big|\, \mathcal{G}_s\big] = \exp -\big\langle X_s, v(t-s,\cdot)\big\rangle,$$

*where* $v$ *is the unique non-negative solution of (1.7). We call* $v$ *the* dual function *of the catalytic super-Brownian motion* $X$.

Thanks to this construction, we can think heuristically of catalytic super-Brownian motion as clouds of uncountably many, infinitesimally small particles which perform catalytic branching Brownian motion at an infinite branching rate whenever the particles hit the support of the catalytic measure.

The class of catalytic super-Brownian motion with uniformly non-polar catalysts $\sigma$ which we deal with in this thesis was introduced by DELMAS [De96]. Other authors have imposed *different* conditions on the catalytic measure. For example DYNKIN [Dy91] treats the class of catalysts whose branching functionals have finite exponential moments of all orders. The main reason to impose any condition on the catalytic measure is to ensure the existence of a *unique* non-negative solution of equation (1.7) and therefore to obtain a characterization of catalytic super-Brownian motion in terms of its log-Laplace equation. Whereas DYNKIN uses a generalized version of the *Gronwall Lemma* to show the uniqueness, DELMAS proposed using *small* test functions $\varphi$: let $T > 0$ and $\varphi \in \mathcal{B}_+^b(\mathbb{R}^d)$ such that $2a_T\|\varphi\|_\infty < 1$. Notice that the condition of uniformly non-polarness on the catalytic measure ensures the finiteness of the constant $a_T$ (see Lemma 2.18). Moreover, let $v$ and $w$ both be non-negative solutions of (1.7). Then clearly both $v$ and $w$ are bounded by $\|\varphi\|_\infty$. Hence,

$$\begin{aligned}\|v-w\|_\infty &\le \sup_{x\in\mathbb{R}^d, t\in[0,T]} \mathbb{E}_x\Big[\int_0^t dA_s \,\big|v^2(t-s,W_s) - w^2(t-s,W_s)\big|\Big]\\ &\le 2a_T\|\varphi\|_\infty\|v-w\|_\infty,\end{aligned}$$

and therefore $v = w$ on $[0,T]\times\mathbb{R}^d$. In this thesis, we are going to make use of this argument in various situations.

We shall also use *non-catalytic* superprocesses with general underlying motion processes of which we would like to remind the reader now (see e.g. [LG99, Chapter II] for more details). Let $\xi = (\xi_t, t \ge 0)$ be a Markov process with values in some Polish space $E$. We also assume that the paths $t \mapsto \xi_t$ of $\xi$ are càdlàg (right-continuous with left limits). Let us write $\mathbb{P}_x^\xi$ for the law of $\xi$ starting in $\xi_0 = x \in E$. Let $\mathcal{M}_f(E)$ denote the space of finite measures on $E$ endowed with the topology of weak convergence. There is a continuous measure valued branching process $U = (U_t, t \geq 0)$ called the *(non-catalytic) superprocess* with spatial motion $\xi$, such that for all $\varphi \in \mathcal{B}_b^+(\mathbb{R}^d)$ and all $s < t$,

$$\mathbb{E}_\nu^U\left[\exp - \langle U_t, \varphi\rangle \,\middle|\, \mathcal{H}_s\right] = \exp - \big\langle U_s, u(t-s,\cdot)\big\rangle,$$

where $u$ is the unique non-negative solution of

$$u(t,x) + \mathbb{E}_x\left[\int_0^t ds\, u^2(t-s,\xi_s)\right] = \mathbb{E}_x[\varphi(\xi_t)], \tag{1.8}$$

and where we denote by $\mathbb{P}_\nu^U$ the law of $U$ starting in $U_0 = \nu \in \mathcal{M}_f(E)$ and by $(\mathcal{H}_t, t \geq 0)$ the $\mathbb{P}_\nu^U$-completed filtration generated by $U$. Notice that equation (1.8) is the integral form of (1.1) choosing the branching rate constant equal to 1. We write $u$ to be the *dual function* of the superprocess $U$. We also speak of $U$ as the *quadratic* superprocess with underlying motion $\xi$. Here, the 'quadratic' refers to the function $\psi(u) = u^2$ in the integral term on the left-hand-side of equation 1.8.

Moreover, if $t \mapsto \xi_t$ is continuous in probability under $\mathbb{P}_x^\xi$ for *every* $x \in E$, then for every $\nu \in \mathcal{M}_f(E)$ the measure valued paths $t \mapsto U_t$ are also continuous in probability under $\mathbb{P}_\nu^U$ (see e.g. [LG99, Proposition II.8]). In particular, we can choose a measurable modification of the process $U$ meaning that the map $(t,\omega) \mapsto U_t(\omega)$ is measurable. Hence, integrals of the form $\int_0^\infty ds\, U_s$ are well defined under $\mathbb{P}_\nu^U$.

In general, it is possible to consider a large class of branching mechanisms instead of the quadratic non-linearity $\psi(u) = u^2$ (see e.g. [LG99]). Indeed, we could replace the quadratic branching in (1.7) or (1.8) by a function $\psi$ of the form

$$\psi(u) = \alpha u + \beta u^2 + \int \pi(dr)\left(\mathrm{e}^{-ru} - 1 + ru\right), \tag{1.9}$$

where $\alpha \geq 0, \beta \geq 0$ and $\pi$ is a sigma-finite measure on $(0,\infty)$ such that

$$\int \pi(dr)\,(r \wedge r^2) < \infty.$$

Notice that there is a one-to-one correspondence between functions of the form (1.9) and *continuous state branching processes* (see e.g. [DLG02, Vo01]). In the non-catalytic case, the total mass process of a superprocess with branching mechanism $\psi$ is given by the continuous state branching process associated to $\psi$.

It is also worth mentioning that in a branching Brownian motion approximation *all* critical branching mechanisms with finite variance lead to quadratic branching $\beta u^2$ in the limit (for more details see the original paper [La67] or [LG99]). Therefore and for simplicity we restrict ourselves to the quadratic branching mechanism $\psi(u) = u^2$ with $\beta = 1$ throughout this thesis. Nevertheless, most of the results extend immediately to more general branching mechanisms $\psi$ of the form (1.9).

### 1.1.2 Collision local time

Roughly speaking, the branching in catalytic super-Brownian motion is governed by the amount of collisions generic particles collect with the catalyst. Therefore, keeping track of these collisions should give a good insight into understanding the whole process. The *collision local time* between a catalytic superprocess and its catalytic measure is a space-time measure which measures the total amount of collisions between the infinitesimal particles of the process and the catalytic measure. This section is devoted to recalling the notion of collision local time for catalytic super-Brownian motion (see also [De96, EP98]).

To begin with, we fix an appropriate space of test functions. Let

$$H_b^T := \left\{ \psi \in \mathcal{B}_+^b([0,\infty) \times \mathbb{R}^d) : \operatorname{supp} \psi \subseteq [0,T] \times \mathbb{R}^d \right\}$$

and $H_b := \bigcup_{T \geq 0} H_b^T$.

Let $\rho$ be a uniformly non-polar measure on $\mathbb{R}^d$ and let $X$ be a $\sigma$-catalytic super-Brownian motion starting in $X_0 = \eta$. According to [De96], we have the following existence theorem:

**Theorem 1.5 (Existence of the collision local time).**
*There is a (random) measure $\mathcal{L}_{\rho,X}$ called the* collision local time *of $X$ with the measure $\rho$, which is supported on $[0,\infty) \times \operatorname{supp} \rho$, such that for all $\psi \in H_b$, we have $\mathbb{P}_\eta^X$-almost surely*

$$\langle \mathcal{L}_{\rho,X}, \psi \rangle = \lim_{\varepsilon \downarrow 0} \int_0^\infty ds \int \rho(dy) \int X_s(dz)\, p_\varepsilon(y,z)\, \psi(s,y). \tag{1.11}$$

*Moreover, almost surely for all $T \geq 0$, we have $\mathcal{L}_{\rho,X}([0,T) \times \mathbb{R}^d) < \infty$ and the convergence (1.11) also holds in $L^p(\mathbb{P}_\eta^X)$ for any $p \geq 1$.*

In particular, if $\rho = \sigma$, we call the random measure $\mathcal{L}_{\sigma,X}$ the *collision local time* of the $\sigma$-catalytic super-Brownian motion $X$ with its catalyst $\sigma$.

## 1.2 An overview of the results

This section intends to be both, an overview of our main results and a guide through the whole thesis. The Sections 1.2.1, 1.2.2 and 1.2.3 correspond to the Chapters 2, 3 and 4 of the thesis.

In Section 1.2.1 we give a direct construction of a random measure which is equal in law to the collision local time $\mathcal{L}_{\sigma,X}$ of a catalytic super-Brownian motion $X$ with its catalytic measure $\sigma$. Under a regularity assumption on the catalytic measure, we show in Section 1.2.2 how the catalytic super-Brownian motion can be *constructed* deterministically from this measure. Moreover, we use this approach to construct a super-Brownian motion in a bounded domain $D \subseteq \mathbb{R}^d$ with reflecting boundary $\partial D$ which is catalyzed by the surface measure on a proper subset $F_1 \subseteq \partial D$ of its boundary. We then use this construction to give a probabilistic representation of (weak) solutions of non-linear boundary value problems in Section 1.2.3.

The results presented in Section 1.2.1 and 1.2.2 (A) are joint work with PETER MÖRTERS (Bath). Also see [MV03]. The Sections 1.2.2 (B) and 1.2.3 are based on a joint project with JEAN-FRANÇOIS DELMAS (ENPC, Paris) leading to the recent preprint [DV03].

### 1.2.1 Chapter 2: A subordination route to the collision local time

Let $\sigma$ be a fixed catalytic measure, $A$ the branching functional associated to $\sigma$, $X$ be a $\sigma$-catalytic super-Brownian motion and $\mathcal{L}_{\sigma,X}$ the collision local time of $X$ with its catalyst $\sigma$.

Define $\mathcal{S}$ to be the *support* of the branching functional $A$ given by

$$\mathcal{S} := \operatorname{supp} A := \{x \in \mathbb{R}^d : \mathbb{P}_x(R = 0) = 1\},$$

where $R := \inf\{t > 0 : A_t > 0\}$. Roughly speaking, the set $\mathcal{S}$ consists of all points $x$ such that the function $t \mapsto A_t$ increases immediately with probability one. If $D$ denotes the closed support of the measure $\sigma$, then $\mathcal{S} \subseteq D$ but $\mathcal{S} \neq D$ in general ($d \geq 2$). However, we show in Lemma 2.7 that $\mathcal{S}$ agrees with the so called *fine support* of $\sigma$ (see Section 2.1.2 for a precise definition). Therefore we call the set $\mathcal{S}$ the fine support of $\sigma$.

In order to construct a random measure which is equal in law with $\mathcal{L}_{\sigma,X}$ we proceed as follows: denote by

$$A_t^{-1} := \inf\{s > 0 : A_s > t\} = \sup\{s > 0 : A_s \le t\}, \tag{1.12}$$

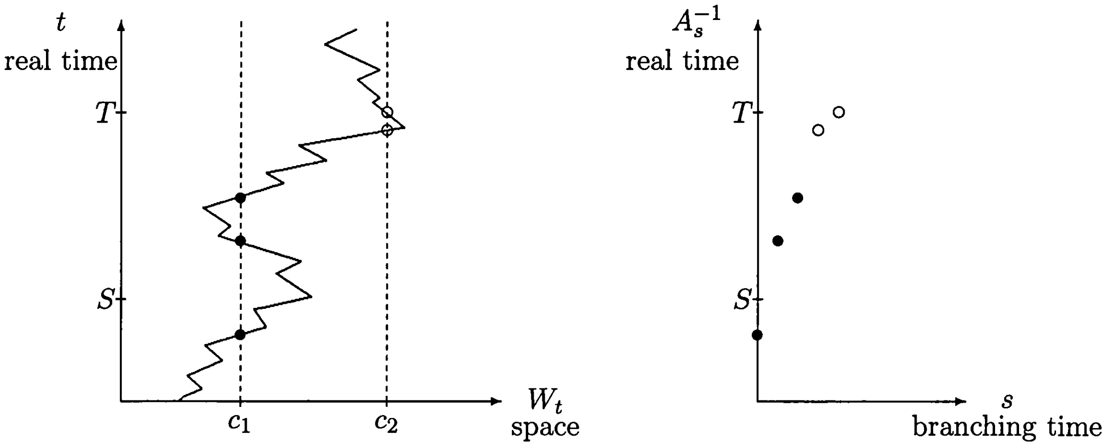
the right continuous inverse of the branching functional $A$, i.e. $A_t^{-1}$ is the first time the $A$ reaches the value $t$. As branching is only possible whenever $A$ increases, doing the time-change $t \mapsto A_t^{-1}$ changes the *real-time scale* into *branching time scale.* Hence, $W \circ A_t^{-1}$ runs the Brownian motion $W$ in branching time scale. In other words, $W \circ A_t^{-1}$ is the position of the Brownian motion at the real time when the branching functional takes the value $t$ for the first time.

Let us define the (time homogenous) Markov process $\xi = (\xi_t, t \ge 0)$ with state space $E = [0, \infty) \times \mathcal{S}$ started at $(r, x) \in E$ by

$$\xi_t := (A_t^{-1} + r, W \circ A_t^{-1}), \qquad \text{for } t \in [0, A_\infty), \tag{1.13}$$

where $W$ is a Brownian motion started in $x$, i.e. the first component of $\xi$ keeps track of the branching time and the second component runs the Brownian motion $W$ in the branching time scale. Notice that $W \circ A_t^{-1}$ is always an element of $\mathcal{S}$ (see Lemma 2.9). Roughly speaking, $t \mapsto W \circ A_t^{-1}$ is the trace of the Brownian motion on the set $\mathcal{S}$.

**Example.** *(Two point catalyst in d=1)* In order to get a better understanding of the process $\xi$, we consider the following example: Let $d = 1$ and $\sigma = \delta_{c_1} + \delta_{c_2}$ for some $c_1, c_2 \in \mathbb{R}$ with $c_1 \ne c_2$. Then $\mathcal{S} = \{c_1, c_2\}$ and according to example (i) after Definition 1.1 we have $A_t = \ell_t^{c_1} + \ell_t^{c_2}$. The following picture illustrates the process $\xi$:




The left hand side of the picture shows a 'path' $t \mapsto W_t$ of a 1-dimensional Brownian motion $W$ run in *real time* and its 'collisions' with $c_1$ marked by • and with $c_2$ marked by $\circ$. The right hand side, illustrates the corresponding path $s \mapsto \xi_s$ of the process $\xi$ run in *branching time.* Here, the vertical axis shows the first component of $\xi$, i.e. the real time $A_s^{-1}$ of the particle at branching time $s$. As the second component $W \circ A_s^{-1}$ of $\xi$ is always an element of $\mathcal{S} = \{c_1, c_2\}$, we describe $W \circ A_s^{-1} = c_1$ by a filled circle $\bullet$ and $W \circ A_s^{-1} = c_2$ by $\circ$. Moreover, looking at the picture, we make the following heuristic observation: for a non-empty catalyst region $C \subseteq \{c_1, c_2\}$ and any time interval $[S, T]$, we can 'measure' the collisions of $W$ with $C$ during the time interval $[S, T]$ *either* by running the Brownian motion in real time (left hand side of the picture) *or* by running the process $\xi$ (right hand side).

Note, that the process $t \mapsto W \circ A_t^{-1}$ inherits the Markov property from the Brownian motion, but $t \mapsto A_t^{-1}$ is *not* Markovian, as its evolution from a given time point also depends on the space position of the Brownian motion and, heuristically, the path $(A_s^{-1}, s \leq t)$ contains the information about the position $W \circ A_t^{-1}$. Combining $A_t^{-1}$ and $W \circ A_t^{-1}$ to the two parameter process $\xi$ we again obtain a strong Markov process. Moreover, it is easy to verify that $t \mapsto \xi_t$ is continuous in probability under $\mathbb{P}_x^{\xi}$ for any $x \in E$ (see Proposition 2.11).

For $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ let $\nu_\eta$ be the law of the first hitting time and place of $\mathcal{S}$ by a Brownian motion distributed according to $\eta$ at time 0, i.e. define

$$\langle \nu_\eta, \psi \rangle = \int \eta(dx)\, \mathbb{E}_x\big[\mathbf{1}_{\{\tau_{\mathcal{S}} < \infty\}} \psi(\tau_{\mathcal{S}}, W_{\tau_{\mathcal{S}}})\big],$$

where $\tau_{\mathcal{S}} := \inf\{t > 0 : W_t \in \mathcal{S}\}$ is the first hitting time of $\mathcal{S}$.

The main idea for the representation of $\mathcal{L}_{\sigma,X}$ is the following: instead of running generic particles in real time and measuring their collisions with the catalyst, we run them in *branching time*, i.e. we consider $U = (U_s, s \geq 0)$ to be a quadratic superprocess with underlying motion $\xi$. Roughly speaking, the measure $U_s$ keeps book of the *real time* and the *real time position* of all particles which have accumulated branching time $s$. Hence, the total occupation measure $\int_0^\infty ds\, U_s$ is a good candidate to have the same law as the collision local time.

It is important to remark, that the measure $\int_0^\infty ds\, U_s$ is constructed *without* any reference to the catalytic super-Brownian motion $X$.

**Theorem 1.6 (Representation of $\mathcal{L}_{\sigma,X}$).**
*Let $U$ be the superprocess with spatial motion $\xi$ starting in $U_0 = \nu_\eta$. Then the law of*

*the total occupation measure of U defined by*

$$\Gamma_\sigma(dr\,dx) := \int_0^\infty ds\, U_s(dr\,dx)$$

*agrees with the law of the collision local time $\mathcal{L}_{\sigma,X}$ of a $\sigma$-catalytic super-Brownian motion $X$ started in $X_0 = \eta$ with its catalytic measure $\sigma$.*

The case of a *one point catalyst* $\sigma = \delta_x$ in dimension $d = 1$ is already treated in [FLG95]. In this case the continuous additive functional $A$ is the local time $\ell^x$ of the Brownian motion $W$ at the catalytic point $x$, and $A^{-1}$ is distributed as a $(1/2)$-stable subordinator, see e.g., [RY99, Chapter VI]. Of course, $W \circ A^{-1} \equiv x$ and the spatial component is redundant in this case. It becomes apparent that the statement $\mathcal{L}_{\sigma,X} = \Gamma_\sigma$ in law is a superprocess analogue of Lévy's well-known result that the local time measure has the same distribution as the occupation measure of a $(1/2)$-stable subordinator.

### 1.2.2 Chapter 3: From collision local time to catalytic super-Brownian motion

Having constructed the collision local time $\Gamma_\sigma$ *without* any reference to $X$, it is natural to ask whether the catalytic super-Brownian motion is already determined by this measure, i.e. whether the full catalytic super-Brownian motion can be recovered deterministically from the collision local time, roughly speaking, by adding the paths of particles between their birth and their death to the picture. Because of the infinite branching intensity at the catalyst, it is the philosophy that $\Gamma_\sigma$ describes the space-time points where branching occurs, hence where particles are born. Particles are released at these points and, given $\Gamma_\sigma$, they move independently according to Brownian excursions from $\mathcal{S}$. By a law of large numbers effect, outside the catalyst, one can only observe the average over these particles, i.e. a heat flow with boundary conditions given by $\Gamma_\sigma$.

This philosophy turns out to be accurate under an additional regularity condition on the catalytic measure. The condition relates the catalytic measure to the capacitary measure of its fine support, and allows us to use last exit decompositions and thus the tool of probabilistic potential theory. In Theorem 1.7 we show that under this condition the catalytic super-Brownian motion can be represented as the solution to the heat equation in the complement of the fine support of the catalytic measure with noisy boundary conditions given by the collision local time. Our result gives an *independent construction* of the catalytic super-Brownian motion.

In a closely related work, [De96] shows that outside the support $\mathcal{S}$ of the catalytic measure, catalytic super-Brownian motion is always determined by the collision local time between the catalytic super-Brownian motion and a *different* measure, the capacitary measure of $\mathcal{S}$. However, we do not know any natural construction of this collision local time, which does not refer to the catalytic super-Brownian motion. Therefore his result does *not* provide an independent construction of the catalytic super-Brownian motion.

In [DFM02] a similar approach is used to represent catalytic super-Brownian motion directly, using an exit-measure construction and the (non-catalytic) superprocess we use in our representation of the collision local time. This method is presented for the case of a stable random measure acting as a catalyst, and is limited to the special case that the catalyst is dense, or more precisely, that the additive functional of Brownian motion induced by the catalytic measure is *strictly* increasing. The paper focuses on applications of this representation to the clumping behavior of the catalytic super-Brownian motion.

An interesting extension of the ideas of [DFM02] to the construction of a more general class of catalytic superprocesses is given in [Kl02]. This representation uses a superprocess with a path-valued Markov process as underlying motion, an approach which is quite different from ours.

### (A) Construction of $\sigma$-catalytic super-Brownian motion in $\mathbb{R}^d$

Let us fix a catalytic measure $\sigma$ and assume that its fine support $\mathcal{S}$ has vanishing Lebesgue measure. The main idea behind our construction is the following: as all branching activity take place only whilst generic particles hit the set $\mathcal{S}$, we expect the mass flow outside $\mathcal{S}$ to behave essentially like heat flow. Hence, the only probabilistic input to catalytic super-Brownian motion is its behaviour *on* the catalyst $\mathcal{S}$, which is given by the collision local time $\Gamma_\sigma$. Therefore, we are able to construct $X$ deterministically from the measure $\Gamma_\sigma$.

An important tool is the *last-exit decomposition* of the Brownian transition semigroup, of which we would like to remind the reader now. We follow [Ma75] (for more details we refer to Section 3.1.3).

Let $\mu$ be the law of the first point in $\mathcal{S}$ visited by the Brownian motion $W$, if $W$ is originally distributed according to Lebesgue measure and also killed at rate 1, i.e.

$$\langle \mu, \psi \rangle = \int_{\mathbb{R}^d} dx \, \mathbb{E}_x \big[ \mathbf{1}_{\{\tau_{\mathcal{S}} < \infty\}} \, \mathrm{e}^{-\tau_{\mathcal{S}}} \ \psi(W_{\tau_{\mathcal{S}}}) \big], \tag{1.16}$$

where $\tau_{\mathcal{S}}$ is the first hitting time of $\mathcal{S}$. The measure $\mu$ is called the *capacitary measure* of $\mathcal{S}$ (see e.g. [Be96, Chapter II]). The associated additive functional $L = (L_t, t \geq 0)$ is called the *capacitary local time* on $\mathcal{S}$. There exists a family of sigma-finite measures $H = (H^x, x \in \mathcal{S})$ on the space of excursions $\omega$ such that, for all $x \in \mathbb{R}^d$ and $\varphi \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\mathbb{E}_x[\varphi(W_t)] = Q_t\varphi(x) + \mathbb{E}_x \Big[ \int_0^\infty \mathbf{1}_{\{s<t\}} H^{W_s}[\varphi(\omega_{t-s})] \, dL_s \Big], \tag{1.17}$$

where $(Q_t, t \geq 0)$ denotes the semigroup of Brownian motion killed on $\mathcal{S}$, given by

$$Q_t\varphi(x) := \mathbb{E}_x \big[ \mathbf{1}_{\{t<\tau_{\mathcal{S}}\}} \varphi(W_t) \big].$$

Roughly speaking $H^x$ is the 'law' of a Brownian excursion from $\mathcal{S}$ starting in $x$. If we denote the length of the excursion $\omega$ by $\zeta(\omega) \in (0, \infty]$, then we let $\varphi(\omega_t) = 0$ for all $t \geq \zeta(\omega)$ on the right hand side of (1.17). Still following [Ma75], the pair $(L, H)$ is called the *exit system* associated to the set $\mathcal{S}$ and the Brownian motion $W$.

Intuitively, equation (1.17) separates the computation of the transition probabilities of Brownian motion in two parts: the first arises from paths that have *not* yet hit $\mathcal{S}$ at time $t > 0$ and the second from excursions starting at the *last-exit-time* from $\mathcal{S}$ which is distributed according to the measure $dL$.

If the capacitary measure $\mu$ on $\mathcal{S}$ is uniformly non-polar, [De96] gives a representation of $\sigma$-catalytic super-Brownian motion which can be seen as a superprocess analogue of the last exit decomposition (1.17). Using the collision local time $\mathcal{L}_{\mu,X}$ of the $\sigma$-catalytic super-Brownian motion $X$ with the capacitary measure $\mu$ of the set $\mathcal{S}$, Delmas proved that the measure valued process

$$\langle Z_t, \varphi \rangle = \langle \eta, Q_t\varphi \rangle + \iint \mathcal{L}_{\mu,X}(ds\,dx) \, \mathbf{1}_{\{s<t\}} \, H^x[\varphi(\omega_{t-s})], \tag{1.19}$$

is indeed a $\sigma$-catalytic super-Brownian motion starting in $Z_0 = \eta$. Nevertheless, this result only gives a representation *not* a construction of the catalytic super-Brownian motion as the construction of the collision local time $\mathcal{L}_{\mu,X}$ according to Theorem 1.5 already *involves* the catalytic super-Brownian motion $X$.

Our aim is to replace $\mathcal{L}_{\mu,X}$ by the random measure $\Gamma_\sigma$, which is constructed *only* using the catalytic measure $\sigma$. As in general the continuous additive functionals $A$ and $L$ do *not* agree, we have to assume in the following that $L$ is absolutely continuous with respect to $A$, or more precisely that there is a density $f$ such that $\mu(dx) = f(x)\,\sigma(dx)$. Then we can replace $\mathcal{L}_{\mu,X}(ds\,dx)$ by $\Gamma_\sigma(ds\,dx)\,f(x)$ in (1.19) to provide an alternative *construction* of the $\sigma$-catalytic process from the measure $\Gamma_\sigma$.

**Theorem 1.7 (Construction of $\sigma$-catalytic super-Brownian motion).**
*Assume that $\sigma$ is a uniformly non-polar measure such that its fine support $\mathcal{S}$ has vanishing Lebesgue measure. Moreover, let the capacitary measure $\mu$ of $\mathcal{S}$ be absolutely continuous with respect to $\sigma$, i.e. there is a density $f$ such that $\mu(dx) = f(x)\sigma(dx)$.*

*(a) The measure valued process $Z = (Z_t, t \geq 0)$ defined by $Z_0 := \eta \in \mathcal{M}_f(\mathbb{R}^d)$ and*

$$\langle Z_t, \varphi\rangle = \langle \eta, Q_t\varphi\rangle + \iint \Gamma_\sigma(ds\,dx)\,\mathbf{1}_{\{s<t\}} f(x)\,H^x[\varphi(\omega_{t-s})], \qquad (1.20)$$

*for any $\varphi \in \mathcal{B}^b_+(\mathbb{R}^d)$ and all $t > 0$, is a $\sigma$-catalytic super-Brownian motion $X$ with start in $X_0 = \eta$.*

*(b) The collision local time $\mathcal{L}_{\sigma,Z}$ of $Z$ with the catalytic measure $\sigma$ is given by the random measure $\Gamma_\sigma$.*

For an *intuitive* interpretation of Theorem 1.7, notice that the first term on the right hand side of (1.20) corresponds to the contribution of those particles that have *not* yet reached the catalyst at time $t$. To understand the second term, interpret $\Gamma_\sigma(ds\,dx)$ as the reactant mass present at time $s$ at the catalyst point $x$. The term $f(x)\Gamma_\sigma(ds\,dx)$ now represents the mass of branching particles born at the space-time point $(s, x)$. The smaller $f(x)$, the more likely it is that particles released from $x$ are killed instantaneously. Particles born at time $s$ in $x$ move away from the catalyst according to the excursion laws $H^x$. As all these particles are independent we can only observe the average effect, and hence, the *only* probabilistic input into the catalytic super-Brownian motion is the collision local time.

Although (1.20) can be seen as the superprocess analogue to the last exit formula (1.17) in a *one* particle picture (note that $dL_s = f(W_s)dA_s$), it is important to observe that (1.20) gives a construction of the stochastic process itself and *not only* of the corresponding transition semigroup as in the case in (1.17).

The results of Theorem 1.7 extend those of [FLG95] for the special case of a one-point catalyst in $d = 1$. In this situation the space component of the collision local time degenerates and explicit calculations can be performed.

If $\mathcal{S}$ has positive Lebesgue measure, our representation holds only on the set $\mathbb{R}^d \setminus \mathcal{S}$. However, if $\varphi$ is supported by a subset of $\mathcal{S}$, a representation of $\langle Z_t, \varphi\rangle$ can be obtained via *exit measures* of the superprocess $U$ as in [DFM02]. This idea has been used by [FX03] to study an interface problem.

The arguments of [De96, Théorème 8.1] show that almost surely $Z$ has a density with respect to Lebesgue measure, which is a smooth solution of the *heat equation* with singular boundary conditions given by the collision local time $\Gamma_\sigma$:

**Corollary 1.8 (Smoothness of the density field).**
*Almost surely, for all $t > 0$ the random measure $Z_t$ on $\mathcal{S}^c := \mathbb{R}^d \setminus \mathcal{S}$ has a density $z_t(y)$ with respect to the Lebesgue measure. Moreover, $z \in \mathcal{C}^\infty\big((0,\infty) \times \mathcal{S}^c\big)$ and it solves the heat equation outside the catalyst, i.e.*

$$\partial_t\, z_t(y) \;=\; \tfrac{1}{2}\Delta_y\, z_t(y) \qquad \textit{for all } t \in (0,\infty),\ y \in \mathcal{S}^c.$$

**Remark 1.9.** It follows immediate from the continuity of the super-Brownian motion $Z$ in the weak topology that for all $\varphi \in \mathcal{C}_+^b(\mathbb{R}^d)$,

$$\lim_{t\downarrow 0} \int dy\, \varphi(y) z_t(y) = \int \eta(dy)\, \varphi(y).$$

Moreover, Theorem 1.7 (b) implies that for all $\psi \in H_b$,

$$\lim_{\varepsilon\downarrow 0} \int_0^\infty dt \int \sigma(dy) \int dx\, z_t(x) p_\varepsilon(x,y) \psi(t,y) = \iint \Gamma_\sigma(dr\, dx)\, \psi(r,x).$$

In other words, the density field $z$ of the catalytic super-Brownian motion $Z$ solves the heat equation on $(0,\infty) \times \mathcal{S}^c$ with generalized initial condition $\eta$ on $\{t = 0\}$ and generalized boundary condition on $(0,\infty) \times \mathcal{S}$ given by the collision local time $\Gamma_\sigma$.

The abstract condition that the capacitary measure $\mu$ on $\mathcal{S}$ is absolutely continuous with respect to the catalytic measure $\sigma$ has to be checked separately for each class of catalysts. We give some examples in Section 3.2.4:

(i) For any *finite* point measure $\sigma \in \mathcal{M}_f^p(\mathbb{R})$ in one dimension, it is easy to see that $\mu$ is absolutely continuous with respect to $\sigma$. See Proposition 3.11.

(ii) If the catalytic measure $\sigma$ is the surface measure on a smooth $d-1$ dimensional manifold $\mathcal{S}$ in $\mathbb{R}^d$, indeed, the capacitary measure $\mu$ on $\mathcal{S}$ is absolutely continuous with respect to $\sigma$. See Proposition 3.13.

(iii) On the other hand, the condition may fail for rougher cases. For example, if $\sigma$ is the Hausdorff measure on the classical middle-third Cantor set. See Proposition 3.15.

**(B) Catalytic super-Brownian motion in a bounded domain with reflecting boundary**

Let $D$ be a bounded domain, i.e. a connected open subset of $\mathbb{R}^d$, $d \geq 2$, with $\mathcal{C}^3$-boundary $\partial D$. Let $F_1$ and $F_2$ be two relatively open subsets of $\partial D$. We assume that $F_1$ and $F_2$ are non-empty, disjoint and that $\bar{F}_1 \cup \bar{F}_2 = \partial D$. We also assume that the relative boundary of $F_1$ is equal to the relative boundary of $F_2$, and that it is either empty or a $\mathcal{C}^2$-manifold of codimension 2.

It is the idea to construct a super-Brownian motion in $D$, such that $F_1$ is a reflecting, catalytic part of the boundary and $F_2$ is just reflecting.

Let $B = (B_t, t \geq 0)$ be a reflecting Brownian motion in $D$, with normal reflection. With a slight abuse of notation, denote by $\mathbb{P}_x$ the law of $B$ started at $x \in \bar{D}$. Roughly speaking, $B$ behaves like standard Brownian motion inside the domain $D$ and reflects towards the interior each time it reaches the boundary $\partial D$. We refer to Section 3.3 for more details on reflecting Brownian motion.

If we denote by $\bar{p}_t(x, y)$ the transition density of the reflecting Brownian motion, there exists a unique continuous additive functional $\ell = (\ell_t, t \geq 0)$ of $B$ called the *local time* on $\partial D$, such that for every $\varphi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$ and $x \in \bar{D}$,

$$\mathbb{E}_x \left[ \int_0^\infty d\ell_s \, \varphi(s, B_s) \right] = \int_0^\infty ds \int_{\partial D} \sigma(dy) \, \varphi(s, y) p_s(x, y), \tag{1.21}$$

where $\sigma$ is the *surface measure* on $\partial D$. In other words, $\sigma$ is the Revuz-measure of the continuous additive functional $\ell$. Also, notice that $\ell$ can be constructed explicitly by (1.6) (see e.g. [ST62, SU65, Hsu84]).

In this section we construct a catalytic super-Brownian motion in $D$ which is catalyzed by the surface measure $\sigma$ restricted to $F_1$ and underlying motion a reflected Brownian motion $B$ in $D$. To be short, we call this process *$F_1$-catalytic super-Brownian motion in $D$*. This particular class of measure valued Markov processes seems to be untreated in the literature so far.

With slight modifications, we use the same approach as we did for the construction of catalytic super-Brownian motion in $\mathbb{R}^d$ explained in the last section. Firstly, we essentially follow Section 1.2.1 to construct a random measure $\Gamma_{\sigma_1}$ which can be seen as the collision local time of the catalytic super-Brownian motion in $D$ — yet to be constructed — and its catalytic measure $\sigma_1(dx) = \mathbf{1}_{\{x \in F_1\}}\sigma(dx)$. Then we proceed in the spirit of Section 1.2.2 (A) to construct a measure valued time homogenous Markov process: the $F_1$-catalytic super-Brownian motion in $D$ with reflecting boundary.

The continuous additive functional associated to $\sigma_1$ is the local time $\ell^1 = (\ell^1_t, t \geq 0)$ on $F_1$ which is given explicitly by

$$d\ell^1_t = \mathbf{1}_{F_1}(B_t)\, d\ell_t. \tag{1.22}$$

Let $\ell^{1,-1}$ denote the right continuous inverse of $\ell^1$, given by

$$\ell^{1,-1}_t := \inf\{s \geq 0 \,:\, \ell^1_s > t\}.$$

Notice that as $\lim_{s\to\infty} \ell^1_s = \infty$ almost surely (see e.g. [SU65, Theorem 7.2]) we have, almost surely for all $t \geq 0$ that $\ell^{1,-1}_t < \infty$. Define $E = (\mathbb{R}_+ \times F_1)$. In the spirit of Section 1.2.1, we define the $E$-valued right continuous and time homogenous Markov process $\xi = (\xi_t, t \geq 0)$ with start in $(s, x) \in E$ by

$$\xi_t := (\ell^{1,-1}_t + s, B \circ \ell^{1,-1}_t),$$

where $B$ is a reflecting Brownian motion in $D$ starting in $x$. Denote by $U = (U_t, t \geq 0)$ the quadratic, *non-catalytic* superprocess with spatial motion $\xi$. Notice that $U$ is an $\mathcal{M}_f(E)$ valued Markov process and denote by $\mathbb{P}^U_\nu$ its law starting from $\nu \in \mathcal{M}_f(E)$. In analogy to Section 1.2.1, define its total occupation measure $\Gamma_{\sigma_1}$ under $\mathbb{P}^U_\nu$ by

$$\Gamma_{\sigma_1}(dr\, dx) := \int_0^\infty ds\, U_s(dr\, dx). \tag{1.23}$$

Similar to Section 1.2.2, the random measure $\Gamma_{\sigma_1}$ plays the key-rôle in the construction of the $F_1$-catalytic super-Brownian motion.

Denote by $\tau_i$ the first hitting time of $F_i, i = 1, 2$. Let $\eta \in \mathcal{M}_f(\bar{D})$ be a finite measure on $\bar{D}$ and define $\nu_\eta \in \mathcal{M}_f(E)$, to be the first hitting distribution of $F_1$ by $(t, B_t)$ starting from $\delta_0 \otimes \eta$, i.e. for any $\psi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$,

$$\langle \nu_\eta, \psi \rangle = \int \eta(dx)\, \mathbb{E}_x[\mathbf{1}_{\{\tau_1 < \infty\}} \psi(\tau_1, B_{\tau_1})].$$

Let us denote by $\mu$ the capacitary measure of the set $F_1$. Then by Lemma 3.20, the measure $\mu$ is absolutely continuous with respect to the surface measure $\sigma$ restricted to $F_1$, i.e. there exists a density $\rho \in \mathcal{B}_+(\mathbb{R}^d)$, such that

$$\mu(dx) = \rho(x)\sigma_1(dx).$$

Let us denote by $L$ the additive functional of $B$ having Revuz measure $\mu$. The theory of exit-systems also applies to $B$ and $F_1$, and we denote by $H = (H^x, x \in F_1)$ the associated family of excursion measures (for more details see Section 3.1.2).

We define, under $\mathbb{P}^U_{\nu_\eta}$, the $\mathcal{M}_f(\bar{D})$-valued process $Z = (Z_t, t \geq 0)$ by $Z_0 := \eta$ and for $t > 0$,

$$\langle Z_t, \varphi \rangle = \langle \eta, Q_t \varphi \rangle + \iint \Gamma_{\sigma_1}(dr\, dx)\, \mathbf{1}_{\{r<t\}} \rho(x) H^x[\varphi(\omega_{t-r})], \tag{1.24}$$

where $\varphi \in \mathcal{B}_+(\bar{D})$ and $Q_t$ denotes the semigroup of the reflected Brownian motion $B$ killed when it first hits $F_1$, i.e.

$$Q_t\varphi(x) = \mathbb{E}_x[\varphi(B_t)\mathbf{1}_{\{t<\tau_1\}}].$$

**Definition 1.10 ($F_1$-catalytic super-Brownian motion).**
*The measure valued process defined by (1.24) is called the* $F_1$- catalytic super-Brownian motion in $D$ *with reflecting boundary.*

Let us give an intuitive understanding of the measure valued process $Z$ defined by (1.24). The measure $Z_t$ describes a cloud of infinitesimal particles at time $t$. The first summand in (1.24) corresponds to those particles which have not reached the catalyst, $F_1$, at time $t$ and which are distributed according to the starting measure $\eta$ at time 0. The second corresponds to the particles which have reached the catalyst before time $t$ and perform a branching process. Particles are released from the catalyst at time $dr$ and location $dx$ according to the random measure $\rho(x)\Gamma_{\sigma_1}(dr\, dx)$ and perform excursions outside the catalyst.

We characterize the finite dimensional marginals of $Z$ in terms of their Laplace transform (Proposition 3.23) and deduce that $Z$ is a time homogeneous Markov process. In Section 1.2.3, we construct a random measure on $F_2$ which can be interpreted as the *exit measure* of the $F_1$-catalytic super-Brownian motion on $F_2$.

### 1.2.3 Chapter 4: Solving non-linear boundary value problems — an application of catalytic super-Brownian motion

Starting with KAKUTANIS pioneering work [Ka44] to express the solution of the classical Dirichlet problem using Brownian motion, much work has been done over the last decades to tackle more difficult boundary value problems. In particular, non-linear equations have been studied using branching and superprocesses. In this section, we give a short overview leading to a boundary value problem with a *mixed* non-linear Neumann condition. We show it is solvable using catalytic super-Brownian motion.

Throughout this section, let $D$ be a bounded domain in $\mathbb{R}^d, d \geq 2$ with $\mathcal{C}^3$-boundary $\partial D$. To start with, consider the classical *Dirichlet problem* (DP),

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \varphi & \text{on } \partial D \end{cases}, \tag{1.25}$$

where $\varphi \in \mathcal{C}(\partial D)$. It is well known that the solution of the DP (1.25) is given in terms of the *exit distribution* of a $d$-dimensional Brownian motion $W = (W_t, t \geq 0)$,

$$u(x) = \mathbb{E}_x\left[\varphi(W_\tau)\right],$$

where $\tau$ is the first hitting time of $\partial D$ by the Brownian motion $W$.

Having solved the classical Dirichlet problem, it is a natural to ask whether Laplace's equation with a *Neumann* boundary condition can also be solved probabilistically. The classical *Neumann problem* (NP) is the following: find a function $u$ such that

$$\begin{cases} \Delta u = 0 & \text{in } D \\ \partial_n u = \varphi & \text{on } \partial D \end{cases}, \tag{1.26}$$

where $\partial_n$ denotes the outward normal derivative and $\varphi \in \mathcal{C}(\partial D)$ is a continuous function on $\partial D$, such that

$$\int_{\partial D} \sigma(dy)\, \varphi(y) = 0, \tag{1.27}$$

where $\sigma$ denotes the surface measure on $\partial D$. Suppose that $u$ is a solution of the NP

(1.26), then the divergence theorem implies that,

$$0 = \int_D dx\, \Delta u(x) = \int_{\partial D} \sigma(dy)\, \partial_n u(y) = \int_{\partial D} \sigma(dy)\, \varphi(y),$$

which makes the condition (1.27) really necessary. To solve the NP (1.26) consider a reflecting Brownian motion $B = (B_t, t \geq 0)$ in $D$ and its local time $\ell = (\ell_t, t \geq 0)$ on the boundary $\partial D$. Recall from Section 1.2.2.B. that $\ell$ is the continuous additive functional of $B$ having Revuz-measure $\sigma$. Then,

$$u(x) = \tfrac{1}{2}\mathbb{E}_x\Big[\int_0^\infty d\ell_s\, \varphi(B_s)\Big] \tag{1.28}$$

is a solution of the NP (1.26) (see e.g. [Br76, Hsu85]). If we do *not* assume any smoothness of the function $\varphi$, e.g. if we only suppose that $\varphi \in L_\infty(\partial D)$ then — according to [Br76] — the function $u$ defined by (1.28) is still a solution of the NP (1.26) in a *weak* sense. Here, *weak* solution means that for any test function $\phi \in \mathcal{C}^2(\bar{D})$, the function $u$ defined by (1.28) solves the integral equation,

$$\int_D dx\, u(x) \Delta\phi(x) = \int_{\partial D} \sigma(dy)\, \partial_n \phi(y) u(y) - \int_{\partial D} \sigma(dy) \varphi(y) \phi(y).$$

Notice that by Greens second identity every (strong) solution of the NP is also solution in the weak sense. This result has been generalized by Bass and Hsu for bounded Lipschitz domains (see [BH91]).

The relation of the Laplacian $\Delta$ as the generator of the Brownian semigroup and the non-linear operator $Gu = \Delta u - u^2$, which generates the log-Laplace semigroup of super-Brownian motion suggests a relation between the non-linear Dirichlet problem

$$\begin{cases} \Delta u = 4u^2 & \text{in } D \\ u = \varphi & \text{on } \partial D \end{cases}, \tag{1.29}$$

and the (classical) super-Brownian motion. And indeed, Dynkin showed (see [Dy91]) that a solution of (1.29) is given in terms of the *exit measure* $X^{\partial D}$ on $\partial D$ of a quadratic super-Brownian motion in $D$. Intuitively, $X^{\partial D}$ is the spatial distribution of generic particles of the super-Brownian motion *frozen* when they first hit $\partial D$. Many generalizations of this problem have been studied e.g. in a series of papers by Dynkin and Kusnetzov [DK96, DK98] and a very powerful technique, the *Brownian snake* has been introduced by Le Gall to handle such problems (see e.g. [LG93, LG95, LG97, LG99]).

Based on a Brownian snake approach, ABRAHAM and ABRAHAM & DELMAS studied the problem $\Delta u = 4u^2$ in a domain with Neumann boundary conditions. In [Ab00] a mixed Dirichlet, non-linear Neumann boundary condition

$$\begin{cases} \Delta u = 4u^2 & \text{in } D \\ u = f & \text{on } F^\circ \\ \partial_n u = 2g & \text{on } \partial D \setminus F \end{cases}, \tag{1.30}$$

is considered, where $F$ is a closed subset of $\partial D$ and $f, g$ are bounded continuous functions on $F$ and $\partial D \setminus F$ respectively. Here, $F^\circ$ denotes the interior of $F$ relative to $\partial D$. In [AD02] the authors study the boundary condition $\partial_n u + \kappa u = \varphi$ on $\partial D$, where $\kappa$ is a non-negative continuous function on $\partial D$ and $\varphi$ is a non-negative measurable function on $\partial D$.

We now give an overview on our results which are based on a joint work with JEAN-FRANCOIS DELMAS (also see [DV03]). We solve Laplace's equation $\Delta u = 0$ in a bounded domain $D$ with two sets of non-linear boundary conditions. First, we treat the problem with a mixed Dirichlet & non-linear Neumann and then with a mixed Neumann & non-linear Neumann boundary condition.

### (A) A mixed Dirichlet & non-linear Neumann boundary condition

In Section 4.1 we give a probabilistic representation of solutions of the mixed Dirichlet non-linear Neumann boundary value problem (DNP)

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = \varphi & \text{on } F_2 \\ \partial_n u + 2u^2 = 0 & \text{on } F_1 \end{cases}, \tag{1.31}$$

where $F_1$ and $F_2$ two relatively open subsets of $\partial D$ and $\varphi \in \mathcal{C}_+(\bar{F}_2)$. We also assume that $F_1$ and $F_2$ are non-empty, disjoint and that $\bar{F}_1 \cup \bar{F}_2 = \partial D$. Moreover, let the relative boundary of $F_1$ be equal to the relative boundary of $F_2$, which is either empty or a $\mathcal{C}^2$-manifold of codimension 2 we denote it by $\partial F$.

In 1999, DYNKIN asked if it is possible to represent non-negative solutions to the DNP (1.31) using catalytic super-Brownian motion. We shall give an affirmative answer to this question. The main idea is to construct the exit measure $Z^{\mathrm{Dir}}$ of the $F_1$-catalytic super-Brownian motion in $D$ (with reflecting boundary) *directly* using time change techniques. Here the superscript 'Dir' refers to the Dirichlet condition on $F_2$.

Roughly speaking, the exit measure $Z^{\mathrm{Dir}}$ is the spatial distribution of the generic particles of a $F_1$-catalytic super-Brownian motion in $D$ *frozen* when they first hit $F_2$. Hence, in order to construct $Z^{\mathrm{Dir}}$ we have to *kill* generic particles once they hit $F_2$. Recall that $\tau_2$ denotes the first hitting time of $F_2$ by the reflected Brownian motion $B$. Consider the local time $\ell^* = (\ell^*_t, t \geq 0)$ on $F_1$ of $B$ killed on $F_2$. It is defined by

$$d\ell^*_t = \mathbf{1}_{\{t<\tau_2\}} d\ell_t. \tag{1.32}$$

Let $\ell^{*,-1} = (\ell^{*,-1}_t, t \geq 0)$ denote the right continuous inverse of the continuous additive functional $\ell^*$ and let $E = (\mathbb{R}_+ \times F_1) \cup \{\delta\}$, where $\delta$ is a cemetery point. We define the $E$-valued time-homogenous Markov process $\xi = (\xi_t, t \geq 0)$ by

$$\xi_t := \begin{cases} (\ell^{*,-1}_t, B \circ \ell^{*,-1}_t) & \text{if } \ell^{*,-1}_t < \infty \\ \delta \quad \text{otherwise} \end{cases},$$

and $\Gamma^{\mathrm{Dir}}$ to be the total occupation measure of the quadratic (non-catalytic) superprocess $U = (U_s, s \geq t)$ with spatial motion $\xi$. Moreover, for $\eta \in \mathcal{M}_f(\bar{D})$ define $\nu_\eta \in \mathcal{M}_f(\mathbb{R}_+ \times F_1)$, to be the hitting distribution of $\mathbb{R}_+ \times F_1$ by $(t, B_t)$, starting from $\delta_0 \otimes \eta$ and killed on $F_2$, i.e. for any non-negative $\psi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$,

$$\langle \nu_\eta, \psi \rangle = \int \eta(dx)\, \mathbb{E}_x[\mathbf{1}_{\{\tau_1<\tau_2\}} \psi(\tau_1, B_{\tau_1})].$$

Denote by $\mathbb{P}^U_{\nu_\eta}$ the law of the superprocess $U$ starting from $U_0 = \nu_\eta$ and by $(H^x, x \in F_1)$ the family of excursion measures from $F_1$. The key definition to solve the DNP (1.31) is the following:

**Definition 1.11 (The exit measure $Z^{\mathrm{Dir}}$).**
*The random measure $Z^{\mathrm{Dir}}$ on $\bar{F}_2$ defined under $\mathbb{P}^U_{\nu_\eta}$ by: for all $\varphi \in \mathcal{B}_+(\bar{F}_2)$,*

$$\langle Z^{\mathrm{Dir}}, \varphi \rangle = \langle \eta, Q^1(\varphi) \rangle + \iint \Gamma^{\mathrm{Dir}}(dr\, dx)\, \rho(x) H^x[\varphi(\omega_{\tau_2}) \mathbf{1}_{\{\tau_2<\infty\}}], \tag{1.34}$$

*with*

$$Q^1(\varphi)(x) = \mathbb{E}_x[\varphi(B_{\tau_2}) \mathbf{1}_{\{\tau_2 \leq \tau_1\}}].$$

*is called the* exit measure *of the $F_1$-catalytic super-Brownian motion on $F_2$. With a slight abuse of notation let us write $\mathbb{P}^Z_\eta$ for its law.*

Performing a first moment calculation it is easy to check that $Z^{\mathrm{Dir}}$ is an almost surely finite measure (see Lemma 4.3). Let us give an intuitive interpretation of the measure

$Z^{\mathrm{Dir}}$ defined by (1.34). Roughly speaking, the measure describes the death positions of infinitesimal particles when they first reach $F_2$. The first summand corresponds to those particles which reach $F_2$ before $F_1$. The second describes particles, which are released from the catalyst $F_1$ at time $dr$ and position $dx$ according to the random measure $\rho(x)\,\Gamma^{\mathrm{Dir}}(dr\,dx)$, performing independent Brownian excursions outside $F_1$ killed when they first reach $F_2$.

Let $\eta \in \mathcal{M}_f(\bar{D})$ and $\gamma := \sup_{x\in\bar{D}} \mathbb{E}_x[\ell_{\tau_2}] < \infty$ (see Lemma 4.35). We characterize the law $\mathbb{P}^Z_\eta$ of the random measure $Z^{\mathrm{Dir}}$ by its log-Laplace functional (see Lemma 4.4): For any $\varphi \in \mathcal{B}_+(\bar{F}_2)$ such that $2\gamma\|\varphi\|_\infty < 1$, we have

$$\mathbb{E}^Z_\eta\big[\exp -\langle Z^{\mathrm{Dir}}, \varphi\rangle\big] = \exp -\langle \eta, w\rangle,$$

where $(w(x), x \in \bar{D})$ is the unique non-negative solution of the integral equation

$$w(x) + \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\, w^2(B_r)\Big] = \mathbb{E}_x\big[\varphi(B_{\tau_2})\big]. \tag{1.35}$$

Now, fix $\varphi \in \mathcal{C}_+(\bar{F}_2)$ and let

$$w(x) := -\log \mathbb{E}^Z_{\delta_x}\big[\exp -\langle Z^{\mathrm{Dir}}, \varphi\rangle\big] \tag{1.36}$$

Let us rewrite equation (1.35) in the form

$$w(x) = \mathbb{E}_x\big[\varphi(B_{\tau_2})\big] - \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\, w^2(B_r)\Big]. \tag{1.37}$$

We make the following heuristic observation: suppose that we can show that $w$ is harmonic. Then — having the probabilistic solution to the classical DP and the classical NP in mind — the function $w$ seems to be a good candidate to solve the DNP (1.31). Roughly speaking, the first summand on the right-hand-side of (1.37) corresponds to the Dirichlet condition on $F_2$ and the second summand to the Neumann condition on $F_1$. Indeed, we show that this intuition does not fail.

If $u \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ and $u$ fulfills (1.31), then $u$ is called a *strong solution* of (1.31). Assuming additional smoothness of the function $w$ defined by (1.36), we prove the following in Section 4.1.3.

**Proposition 1.12 (Strong solutions).**
*If $w \in \mathcal{C}^1(\bar{D})$ then $w$ defined by (1.36) is the unique strong solution to the DNP (1.31).*

In general, it is *not* clear that $w$ belongs to $\mathcal{C}^1(\bar{D})$. Hence, we can *not* expect strong solutions in general. However, we still obtain that $w$ is a solution of the DNP (1.31) in a *weak* sense (see Section 4.1.4), but it is *not* clear that weak solutions of the DNP (1.31) are still unique. Let us define a set of test functions by

$$S_1 := \Big\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D}); \Delta\phi \text{ is bounded in } D,\ \partial_n\phi = 0 \text{ on } F_1, \phi = 0 \text{ on } F_2\Big\}.$$

A bounded function $u \in \mathcal{B}_+(\bar{D})$ is called a *weak solution* of the mixed Dirichlet non-linear Neumann boundary value problem (DNP) given by (4.14) if $u \in \mathcal{C}(\bar{D})$ and for every test function $\phi \in \mathcal{S}^1$,

$$\int_D dx\, u(x)\, \Delta\phi(x) = \int_{F_2} \sigma(dy)\, \partial_n\phi(y)\varphi(y) + 2\int_{F_1} \sigma(dy)\, \phi(y)u^2(y).$$

Notice, according to Green's second identity, every strong solution is also a weak solution. This indeed motivates this definition.

**Theorem 1.13 (Weak solutions).**
*The function $w$ given by (1.36) is a non-negative weak solution of the DNP (1.31).*

### (B) A mixed Neumann & non-linear Neumann boundary condition

In Section 4.2, we eventually tackle the modified problem, where the Dirichlet condition on $F_2$ in the DNP (1.31) is replaced by a Neumann condition, i.e. we want to solve the mixed Neumann non-linear Neumann boundary value problem (NNP)

$$\begin{cases} \Delta u = 0 & \text{in } D \\ \partial_n u - 2\varphi = 0 & \text{on } F_2 \\ \partial_n u + 2u^2 = 0 & \text{on } F_1 \end{cases}, \tag{1.40}$$

for some continuous non-negative function $\varphi$ on $\bar{F}_2$. Intuitively, the solution would be related to the dual equation of a certain random measure $Z^{\mathrm{Neu}}$ (here the superscript 'Neu' corresponds to the Neumann condition on $F_2$) given by

$$w(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r^1\, w^2(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r^2\, \varphi(B_r)\Big], \tag{1.41}$$

where $\ell^1$ and $\ell^2$ denote the local times on $F_1$ and $F_2$ respectively. However, for general $\varphi \in \mathcal{C}_+(\bar{F}_2)$ the right-hand-side of (1.41) is infinite. To turn around these singularities, we first consider the approximating problem $\theta$-NNP,

$$\begin{cases} \Delta u = 2\theta u & \text{in } D \\ \partial_n u - 2\varphi = 0 & \text{on } F_2 \\ \partial_n u + 2u^2 = 0 & \text{on } F_1 \end{cases} \tag{1.42}$$

for $\theta > 0$ and then we let $\theta$ tend to zero. Although we want to use the $\theta$-NNP as an approximation for the NNP, the study of the $\theta$-NNP is interesting on its own right. Roughly speaking, to treat the $\theta$-NNP probabilistically, we have to introduce a *soft killing* for generic particles. The technique we use to introduce this killing are close to those developed in [AD02] in a different context.

In Section 4.2.1 we construct a family of measures $(Z_\theta^{\text{Neu}}, \theta > 0)$ on $F_2$ — in this thesis called the *Neumann boundary measures* on $F_2$ — such that their log-Laplace transform

$$w_\theta(x) := -\log \mathbb{E}^Z_{\delta_x}\big[\exp -\langle Z_\theta^{\text{Neu}}, \varphi\rangle\big]$$

is a non-negative solution of the integral equation

$$w_\theta(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r^1 \; \mathrm{e}^{-\theta r}\, w_\theta^2(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r^2 \; \mathrm{e}^{-\theta r}\, \varphi(B_r)\Big]. \tag{1.43}$$

Moreover, there is a family of constants $(c_\theta, \theta > 0)$ (see Section 3.3.3 such that

$$\mathbb{E}_x\Big[\int_0^\infty d\ell_r \; \mathrm{e}^{-\theta r}\Big] \leq c_\theta.$$

If $\varphi$ is small enough — in the sense that $2c_\theta^2\|\varphi\|_\infty < 1$ — then $w_\theta$ is the unique non-negative solution of equation (1.43) (see Lemma 4.20). If $w_\theta \in \mathcal{C}^1(\bar{D})$, one shows using the same methods as in Section 4.1.3, that $w_\theta$ is a *strong* solution of the $\theta$-NNP. However, in general it is not clear that $w_\theta \in \mathcal{C}^1(\bar{D})$ and therefore again it is worth to consider *weak* solutions of the $\theta$-NNP. Therefore we define an appropriate space of test functions by

$$S_2 := \Big\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D}) : \Delta\phi \text{ bounded }; \partial_n \phi = 0 \text{ on } \partial D\Big\}.$$

Let $\theta \geq 0$. A function $u \in \mathcal{B}_+(\bar{D})$ is called a *weak solution* of the $\theta$-NNP (the NNP for $\theta = 0$ respectively) if $u \in \mathcal{C}(\bar{D})$ and for all $\phi \in S_2$,

$$\int_D dx\, u(x)\Delta\phi(x) = 2\theta \int_D dx\, u(x)\phi(x) - 2\int_{F_2} \sigma(dy)\, \phi(y)\varphi(y) + 2\int_{F_1} \sigma(dy)\, u^2(y)\phi(y).$$

Again notice that by Greens second identity every strong solution of the $\theta$-NNP is also a solution in the weak sense. The following result about the existence and uniqueness of weak solution of the $\theta$-NNP is proved in Section 4.2.3.

**Theorem 1.14 (Existence and uniqueness of weak solutions of the $\theta$-NNP).** *For every $\theta > 0$ the function $w_\theta$ is a non-negative weak solution of the $\theta$-NNP. If additionally, $2c_\theta^2 \|\varphi\|_\infty < 1$ then $w_\theta$ is the unique non-negative weak solution.*

Recall that we denote by $(\mathcal{F}_t, t \geq 0)$ the filtration generated by the reflecting Brownian motion $B$. The proof of Theorem 1.14 is based on the following *martingale characterization* of weak solutions:

**Proposition 1.15 (Martingale characterization of weak solutions).**
*A non-negative function $u \in \mathcal{C}(\bar{D})$ is a weak solution of the $\theta$-NNP if and only if the process $N = (N_t, t \geq 0)$ defined by*

$$N_t = u(B_t) - u(B_0) - \theta \int_0^t dr\, u(B_r) + \int_0^t d\ell_r \, \left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - u^2(B_r)\mathbf{1}_{F_1}(B_r)\right],$$

*is a continuous $\mathcal{F}_t$-martingale.*

Notice that in the martingale formulation of 'weak solution' we do *not* have to use any test functions. Moreover, the fact that we were able to show an 'if and only if' statement in the last proposition enables us to show *uniqueness* for weak solutions using an argument based on the martingale convergence theorem (see Section 4.2.3 for the details).

To solve the NNP for $\theta = 0$, we notice in Section 4.2.4 that the family of Neumann boundary measures $(Z_\theta^{\mathrm{Neu}}, \theta > 0)$ is an increasing family of measures which converges towards a measure $Z^{\mathrm{Neu}}$ as $\theta \downarrow 0$. We deduce that $(w_\theta, \theta > 0)$ increases to a limit $w$ defined on $\bar{D}$ such that

$$w(x) = -\log \mathbb{E}_{\delta_x}^{Z}\left[\exp -\langle Z^{\mathrm{Neu}}, \varphi\rangle\right].$$

Under an additional technical assumption we prove the following in Section 4.2.4:

**Theorem 1.16 (Weak solution of the NNP).**
*Assume that $\bar{F}_1 \cap \bar{F}_2 = \emptyset$. Then $w$ is a weak solution to the NNP.*

# Chapter 2

# Representation of the collision local time

*(This chapter is based on a joint work*[1] *with Peter Mörters, Bath.)*

In this chapter, we construct a random measure $\Gamma_\sigma$ which is equal in law to the collision local time $\mathcal{L}_{\sigma,X}$ of a $\sigma$-catalytic super-Brownian motion $X$ with its catalyst $\sigma$. The construction is based on a time-change for the generic particles and does *not* refer to the catalytic super-Brownian motion $X$.

The chapter is organized as follows: In Section 2.1 we first review basic properties of continuous additive functionals. Then we associate to a given catalyst $\sigma$ its branching functional. Furthermore, we introduce the fundamental time change on which the construction is based. Then in Section 2.2 we first give a detailed statement of the main result: the representation in law of the collision local time. The remaining part of the section is then devoted to give its proof.

[1] See also [MV03]; submitted to *Stochastic Processes and their Applications.*

## 2.1 Introduction and preparation

In this section we travel along the following plan: firstly, in Section 2.1.1, we review some basic properties of the general theory of continuous additive functionals. In Section 2.1.2 we then apply the general theory to associate to a given catalytic measure $\sigma$ its branching functional $A$. Based on the branching functional we finally introduce in Section 2.1.3 the fundamental time change idea on which the whole construction is based. To start with we can not avoid fixing some notation which we will use throughout this thesis.

**Notation.** If $E$ is a polish space, let $\mathcal{B}(E)$ denote its Borel sigma-field as well as the set of real measurable functions defined on $E$. Let $\mathcal{B}_+(E)$ (resp. $\mathcal{C}(E)$) be the subset of $\mathcal{B}(E)$ of non-negative (resp. continuous) functions. The superscript $b$ added to any function space denotes the corresponding subspace of bounded functions. For $\varphi \in \mathcal{B}^b(E)$, let $\|\varphi\|_\infty := \sup_{x\in E} |\varphi(x)|$. Let $\mathcal{M}(E)$ (resp. $\mathcal{M}_f(E)$) be the set of (finite) measures on $E$, endowed with the topology of weak convergence. For $\nu \in \mathcal{M}(E)$ and $\varphi \in \mathcal{B}(E)$, we use both notations $\langle \nu, \varphi \rangle$ and $\int_E \nu(dx)\ \varphi(x)$ to denote the Lebesgue integral of the function $\varphi$ with respect to the measure $\nu$. For a subset $A \subseteq E$ we write $A^{\mathrm{c}} := E \setminus A$ to denote the complement of $A$.

Denote by $W = (W_t, t \geq 0)$ a $d$-dimensional standard Brownian motion and by $\mathbb{P}_{s,x}$ its law if started at time $s \geq 0$ at $x \in \mathbb{R}^d$. As usual, we write $\mathbb{P}_x$ for $\mathbb{P}_{0,x}$. Moreover, for any measure $\nu \in \mathcal{M}(\mathbb{R}^d)$, we write $\mathbb{P}_\nu$ for the Brownian motion distributed according to $\nu$ at time zero. We always think of $W$ to be the coordinate process $W_t(\omega) := \omega_t$ defined on the canonical Wiener space of continuous functions $\omega \in \Omega = \big(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}, \mathbb{P}_x\big)$. We usually write just $W_t$ instead of $W_t(\omega)$. Moreover, let $(\mathcal{F}_t, t \geq 0)$ be the right continuous filtration generated by $W$ completed the usual way, i.e. $A \in \mathcal{F}_t$ if and only if for each probability measure $\nu$ on $\mathbb{R}^d$ there is a set $A_1$ in the sigma-field generated by $W$ up to time $t$ and a set $A_2 \in \mathcal{F}$ such that their symmetric difference $A \Delta A_1 \subseteq A_2$ and $\mathbb{P}_\nu(A_2) = 0$.

### 2.1.1 Continuous additive functionals

Recall, that a *continuous additive functional* (CAF) of $W$ is an $\mathcal{F}_t$-adapted stochastic process $A = (A_t, t \geq 0)$ such that the paths of $t \mapsto A_t(\omega)$ are non-decreasing, continuous and start from zero except for $\omega \in \Lambda$ where $\mathbb{P}_\nu(\Lambda) = 0$ for all $\nu \in \mathcal{M}(\mathbb{R}^d)$. Moreover, we have for each pair of time points $s, t$ the additivity property

$$A_{s+t} = A_t + A_s \circ \theta_t, \tag{2.1}$$

almost surely with respect to $\mathbb{P}_\nu$ for every $\nu \in \mathcal{M}(\mathbb{R}^d)$. Here, $(\theta_t, t > 0)$ is the family of *shift operators* defined on $\Omega$ by $\theta_t \omega_s = \omega_{t+s}$ for all $s \geq 0$. The effect of $\theta_t$ to a path $\omega$ is to cut off the part of the path before time $t$ and *shift* the remaining part in time. Also recall, that a random variable $T : \Omega \to [0, \infty]$ is called a *$\mathcal{F}_t$-stopping time* if $\{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$. Moreover let

$$\mathcal{F}(T) := \{A \in \mathcal{A} : A \cap \{T < t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

From the general theory of continuous additive functionals (see e.g. [BlG68, Chapter IV]) it is well known, that $A$ also satisfies the *strong* additivity property, i.e. for any $\mathcal{F}_t$-stopping time $T$, we have almost surely for all $t \geq 0$,

$$A_{t+T} = A_T + A_t \circ \theta_T. \tag{2.2}$$

Define the random variable $R := \inf\{t > 0 : A_t > 0\}$ with the usual convention that $\inf \emptyset = +\infty$. Moreover, we define the *support* of the continuous additive functional $A$ to be

$$\mathcal{S} := \operatorname{supp} A := \{x \in \mathbb{R}^d : \mathbb{P}_x(R = 0) = 1\},$$

i.e. intuitively, $\mathcal{S}$ consist of those points $x$, such that, for $W$ starting from $x$, the function $t \mapsto A_t$ increases immediately with probability one. Let $\tau_{\mathcal{S}}$ be the first hitting time of $\mathcal{S}$, i.e.

$$\tau_{\mathcal{S}} := \inf\{t > 0 : W_t \in \mathcal{S}\}.$$

A point $x \in \mathbb{R}^d$ is called *regular* for $\mathcal{S}$, if $\mathbb{P}_x(\tau_{\mathcal{S}} = 0) = 1$. If every point $x \in \mathcal{S}$ is regular for $\mathcal{S}$, then we call $\mathcal{S}$ *regular for itself.* The following is Proposition 3.5 of Chapter V in [BlG68].

**Proposition 2.1 (Regularity of $\mathcal{S}$).**
*We have $\tau_{\mathcal{S}} = R$ almost surely on $\{R < \infty\} = \{\tau_{\mathcal{S}} < \infty\}$ and $\mathcal{S}$ is regular for itself.*

It is an interesting question to analyze the set on which the random function $t \mapsto A_t$ increases. We define the points of *right-increase* of $A$ by

$$M^R := \{s \geq 0 : A_{s+\varepsilon} > A_s \text{ for all } \varepsilon > 0\},$$

and the points of *increase* of $A$ by

$$M^I := \{s \geq 0 : A_{s+\varepsilon} > A_{s-\varepsilon} \text{ for all } \varepsilon > 0\}.$$

The following Proposition (which is Theorem 3.8 in Chapter V of [BlG68]) gives an affirmative answer to the question of analyzing the points of increase:

**Proposition 2.2 (The points of increase of $t \mapsto A_t$).**
*We have almost surely $M^R \subseteq \{t \geq 0 : W_t \in \mathcal{S}\} \subseteq M^I$ and almost surely for all $t \geq 0$,*

$$A_t = \int_0^t dA_s \, \mathbf{1}_{\mathcal{S}}(W_s).$$

During the sixties and seventies of the last century, it turned out that continuous additive functionals are closely related to a class of deterministic functions — well known from classical potential theory — of which we would like to remind the reader now. Recall that a function $\psi \in \mathcal{B}(\mathbb{R}^d)$ is called *$\alpha$-excessive* if

(i) for all $t > 0$, we have $\mathrm{e}^{-\alpha t} \, \mathbb{E}_x[\psi(W_t)] \leq \psi(x)$ for all $x \in \mathbb{R}^d$,

(ii) and $\lim_{t \downarrow 0} \mathrm{e}^{-\alpha t} \, \mathbb{E}_x[\psi(W_t)] = \psi(x)$ for all $x \in \mathbb{R}^d$.

For $\psi \in \mathcal{B}(\mathbb{R}^d)$, any $\mathcal{F}_t$-stopping time $T$ and all $x \in \mathbb{R}^d$ define

$$Q_T\psi(x) := \mathbb{E}_x\big[\mathbf{1}_{\{T<\infty\}} \, \mathrm{e}^{-\alpha T} \, \psi(W_T)\big].$$

A function $\psi \in \mathcal{B}(\mathbb{R}^d)$ is called a *regular $\alpha$-potential* if $\psi$ is $\alpha$-excessive and for every increasing sequence of $\mathcal{F}_t$-stopping times $(T_n)$ with limit $T$, we have

$$\lim_{n\to\infty} Q_{T_n}\psi(x) = Q_T\psi(x), \tag{2.3}$$

for all $x \in \mathbb{R}^d$. It is not difficult to show (see e.g. [Bl92, Chapter VII.2]) that for any continuous additive functional $A$, assuming that the function

$$x \mapsto \mathbb{E}_x\Big[\int_0^\infty dA_s \; \mathrm{e}^{-\alpha s}\Big]$$

is finite for all $x$, it is a regular $\alpha$-potential. The following remarkable theorem states that even the converse is true (for a proof of this result see e.g. [Bl92, Theorem VII.2.2]).

**Theorem 2.3 (Volkonski-Sŭr-Meyer).** *If $\psi$ is a regular $\alpha$-potential, then there is* *a unique continuous additive functional $A = (A_t, t \geq 0)$ such that for all $x \in \mathbb{R}^d$,*

$$\psi(x) = \mathbb{E}_x\Big[\int_0^\infty dA_s \, \mathrm{e}^{-\alpha s}\Big].$$

In the next section we are going to apply these results to branching functionals we are interested in.

### 2.1.2 The branching functional associated to $\sigma$

Let $\sigma$ be a uniformly non-polar measure on $\mathbb{R}^d$ which serves as a fixed catalyst throughout this chapter. In this section we are going to introduce a continuous additive functional $A$ of $W$ which is associated to $\sigma$ in a natural way. We refer to $A$ as the *branching functional* associated to $\sigma$.

We start reviewing a preliminary estimate on the catalytic measure (see e.g. [De96, Lemme 2.1]). Recall, that we write $p_s(x, y)$ for the heat kernel.

**Lemma 2.4.** *There is a constant $c > 0$ such that for all $x \in \mathbb{R}^d$ and $s > 0$,*

$$\int \sigma(dy)\, p_s(x, y) \leq c \frac{1}{(s \wedge 1)^{1-\beta}},$$

*where $\beta \in (0, 1)$ is the constant in (1.2) corresponding to $\sigma$. Moreover, we have for all $x, x' \in \mathbb{R}^d$ and all $s > 0$,*

$$\int \sigma(dy)\, \big|p_s(x, y) - p_s(x', y)\big| \leq c(s \wedge 1)^{\beta - 3/2}|x - x'|.$$

**Definition 2.5 (The $\alpha$-potential of $\sigma$).**
*For $\alpha > 0$ define the* $\alpha$-potential *$u_\sigma^\alpha \in \mathcal{B}_+(\mathbb{R}^d)$ of the measure $\sigma$ by*

$$u_\sigma^\alpha(x) := \int \sigma(dy) \int_0^\infty ds \, \mathrm{e}^{-\alpha s}\, p_s(x, y).$$

Notice that we can deduce from Lemma 2.4, that $u_\sigma^\alpha$ is bounded on $\mathbb{R}^d$ for any $\alpha > 0$. We even can prove more, in particular we shall see that the potential $u_\sigma^\alpha$ is a continuous function on $\mathbb{R}^d$:

**Lemma 2.6 (The branching functional associated to $\sigma$).**
*For any $\alpha > 0$, the function $u_\sigma^\alpha$ is a regular $\alpha$-potential. In particular, there is a unique (independent of $\alpha$) continuous additive functional $A = (A_t, t \geq 0)$ of the Brownian*

*motion $W$ such that for all $x \in \mathbb{R}^d$,*

$$\mathbb{E}_x\Big[\int_0^\infty dA_s\, \mathrm{e}^{-\alpha s}\Big] = \int \sigma(dy) \int_0^\infty ds\, \mathrm{e}^{-\alpha s}\, p_s(x,y).$$

*We call $A$ the* branching functional *associated to the catalyst $\sigma$.*

*Proof.* It is easy to see using Fubini's Theorem and the Chapman-Kolmogorov equation that

$$\mathrm{e}^{-\alpha t}\, \mathbb{E}_x[u_\sigma^\alpha(W_t)] = \int_t^\infty ds \int \sigma(dy)\, \mathrm{e}^{-\alpha s}\, p_s(x,y).$$

Hence, $u_\sigma^\alpha$ is $\alpha$-excessive. To verify that $u_\sigma^\alpha$ is a regular $\alpha$-potential, it is enough to show that $x \mapsto u_\sigma^\alpha(x)$ is a continuous function on $\mathbb{R}^d$. For then we have

$$\lim_{n\to\infty} \mathrm{e}^{-\alpha T_n}\, u_\sigma^\alpha(W_{T_n}) = \mathrm{e}^{-\alpha T}\, u_\sigma^\alpha(W_T),$$

$\mathbb{P}_x$-almost surely on $\{T < \infty\}$ by the sample path continuity of the Brownian motion and (2.3) follows using dominated convergence. To prove the continuity define for $\varepsilon > 0$, the function

$$u_{\sigma,\varepsilon}^\alpha(x) := \int_\varepsilon^\infty ds \int \sigma(dy)\; \mathrm{e}^{-\alpha s}\, p_s(x,y),$$

and observe that by the second part of Lemma 2.4,

$$\begin{aligned} |u_{\sigma,\varepsilon}^\alpha(x) - u_{\sigma,\varepsilon}^\alpha(x')| &\le \int_\varepsilon^\infty ds\, \mathrm{e}^{-\alpha s} \int \sigma(dy) |p_s(x,y) - p_s(x',y)| \\ &\le c \int_\varepsilon^\infty ds\; \mathrm{e}^{-\alpha s} (s \wedge 1)^{\beta - 3/2}\, |x - x'|. \end{aligned}$$

Hence, for any $\varepsilon > 0$, the function $u_{\sigma,\varepsilon}^\alpha$ is continuous. Moreover, notice that $u_{\sigma,\varepsilon}^\alpha$ converges uniformly to $u_\sigma^\alpha$ as by the first part of Lemma 2.4, we obtain

$$\begin{aligned} \|u_{\sigma,\varepsilon}^\alpha - u_\sigma^\alpha\|_\infty &\le \sup_{x\in\mathbb{R}^d} \int_0^\varepsilon ds \int \sigma(dy)\; \mathrm{e}^{-\alpha s}\, p_s(x,y) \\ &\le c \int_0^\varepsilon ds\; \mathrm{e}^{-\alpha s}\, \frac{1}{(1 \wedge s)^{1-\beta}}. \end{aligned}$$

The 'in particular statement' now follows by the Volkonskii-Sŭr-Meyer Theorem. Also notice that the continuous additive functional $A$ does *not* depend on $\alpha$ by [BlG68, Theorem IV 2.13]. □

This lemma and [BlG68, Theorem VI.3.1] imply that for all $\varphi \in \mathcal{B}^b_+(\mathbb{R}^d)$,

$$\mathbb{E}_x\Big[\int_0^\infty dA_s\, \mathrm{e}^{-\alpha s}\, \varphi(W_s)\Big] = \int_0^\infty ds \int \sigma(dy)\, p_s(x,y)\, \mathrm{e}^{-\alpha s}\, \varphi(y).$$

Hence a monotone class argument (see e.g. Theorem II.3.2 in [RW00] also noticing that the set of functions of the form $(s,y) \mapsto \mathrm{e}^{-\alpha s}\, \varphi(y)$ is closed under multiplication) imply that for all non-negative, measurable and bounded functions $\varphi \in \mathcal{B}^b_+(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\mathbb{E}_x\Big[\int_0^\infty dA_s\, \varphi(s, W_s)\Big] = \int_0^\infty ds \int \sigma(dy)\, p_s(x,y) \varphi(s,y). \tag{2.4}$$

If for an additive functional $A$ equation (2.4) holds, then $\sigma$ is called the *Revuz-measure* associated to the continuous additive functional $A$.

Having associated the continuous additive functional $A$ to a given uniformly non-polar measure $\sigma$, it is natural to ask whether the support of the additive functional $\operatorname{supp} A$ agrees with the support of $\sigma$? To answer this question, the euclidian topology turns out to be too rough in general and we have to endow $\mathbb{R}^n$ with a topology in which the open sets are determined by the following condition: with probability one, a Brownian path, starting from an arbitrary point of such a set, does not leave this set during a positive time interval.

To be more specific, we first have to introduce a *larger* sigma-field then the usual Borel sets. A set $A \subseteq \mathbb{R}^n$ is called a *nearly Borel set* (relative to the process $W$) if for any initial distribution $\kappa \in \mathcal{M}(\mathbb{R}^d)$ there are Borel sets $B, B' \in \mathcal{B}(\mathbb{R}^d)$ such that $B \subseteq A \subseteq B'$ and $\int \kappa(dx)\, \mathbb{P}_x(W_t \in B' \setminus B \text{ for some } t \geq 0) = 0$. Roughly speaking, a set $A$ is nearly Borel if the Brownian motion $W$ can not distinguish it from a Borel set. Then the class of nearly Borel sets forms a sigma-field denoted by $\mathcal{B}^{\mathrm{n}}(\mathbb{R}^d)$ and we have $\mathcal{B}(\mathbb{R}^d) \subseteq \mathcal{B}^{\mathrm{n}}(\mathbb{R}^d)$.

A set $O$ is called *finely open* if for each $x \in O$ there is a nearly Borel set $B$ such that $O^{\mathrm{c}} \subseteq B$ and $\mathbb{P}_x(\tau_B > 0) = 1$, where $\tau_B$ denotes the first hitting time of the set $B$. By the continuity of Brownian paths every open set is also finely open. Moreover, a set $C$ is called *finely closed* if $C^{\mathrm{c}}$ is finely open. According to [BlG68, Chapter II.4] a Borel set $C$ is finely closed if and only if $C^{\mathrm{r}} \subseteq C$ and the fine closure of a set $C$ is $C \cup C^{\mathrm{r}}$, where $C^{\mathrm{r}}$ denotes the set of regular points for $C$, i.e.

$$C^{\mathrm{r}} := \{x \in \mathbb{R}^d : \mathbb{P}_x(\tau_C = 0) = 1\}.$$

Hence, in particular, we obtain from Proposition 2.1 as $\mathcal{S}$ is regular for itself that $\mathcal{S}$ is finely closed. Also notice that in dimension $d = 1$ the fine topology generated by a linear Brownian motion agrees with the Euclidian topology as all points $x \in \mathbb{R}$ are regular.

Let $\sigma$ be a uniformly non-polar measure. There is a smallest *finely closed* set $F$ such that $\sigma(F^c) = 0$. We call $F$ the *fine support* of $\sigma$.

**Lemma 2.7 (The support of the branching functional).**
*The fine support $F$ of a catalytic measure $\sigma$ agrees with the support $\mathcal{S}$ of the associated branching functional $A$.*

*Proof.* We say that the continuous additive functional $A$ *vanishes* on a set $D \subseteq \mathbb{R}^d$ if

$$\mathbb{E}_x\Big[\int_0^\infty dA_s\, \mathbf{1}_D(W_s)\Big] = 0$$

for all $x \in \mathbb{R}^d$. As

$$\mathbb{E}_x\Big[\int_0^\infty dA_s\, \mathbf{1}_D(W_s)\Big] = \int_D \sigma(dy) \int_0^\infty ds\, p_s(x,y),$$

$A$ vanishes on $D$ if and only if $\sigma(D) = 0$. By [BlG68, Corollary V.3.10] $\operatorname{supp} A$ is the smallest finely closed set on whose complement $A$ vanishes. $\square$

**Remark 2.8.** Notice that by Proposition 2.2 and Lemma 2.7 we see that,

$$A_t = \int_0^t dA_s\, \mathbf{1}_F(W_s)$$

and hence the continuous additive functional $A$ increases only on the set $\{t : W_t \in F\}$.

### 2.1.3 Changing to branching time

In this section we introduce the major idea on which the construction of the collision local time is based: the time change with the inverse of the branching functional.

Define the process $A^{-1} = (A_t^{-1}, t \geq 0)$ to be the right continuous inverse of the continuous additive functional $A$, which is given by

$$A_t^{-1} := \inf\{s > 0 : A_s > t\}. \tag{2.5}$$

Note, that by definition $R = A_0^{-1}$. Hence, we deduce from Proposition 2.1 that for all $x \in \mathbb{R}^d$ we have $\mathbb{P}_x$-almost surely $A_0^{-1} = \tau_{\mathcal{S}}$.

**Lemma 2.9 (Changing time traces $W$ on $\mathcal{S}$).**
*For every $t \geq 0$, the random variable $A_t^{-1}$ is a $\mathcal{F}_t$-stopping time and furthermore we have $W \circ A_t^{-1} \in \mathcal{S}$ almost surely.*

*Proof.* Let $s, t \in \mathbb{R}_+$ and observe that $\{A_t^{-1} < s\} = \{A_s > t\} \in \mathcal{F}_s$. Hence, $A_t^{-1}$ is a $\mathcal{F}_t$-stopping time. Now, let $t > 0$ and we want to show that $W \circ A_t^{-1} \in \mathcal{S}$. First observe that $A \circ A_t^{-1} = t$ and for all $\varepsilon > 0$, we have

$$A \circ [A_t^{-1} + \varepsilon] > t.$$

Hence, almost surely, $A_t^{-1} \in M^R$ and therefore, we deduce $W \circ A_t^{-1} \in \mathcal{S}$ using Proposition 2.2. □

Let $E := \mathbb{R}_+ \times \mathcal{S}$ and define a metric $d_E$ on $E$ by

$$d(\hat{x}_1, \hat{x}_2) := |s_2 - s_1| + |x_2 - x_1|,$$

where as usual $|\cdot|$ denotes both the absolute value in $\mathbb{R}$ and the Euclidian norm on $\mathbb{R}^d$. We define a stochastic process $\xi = (\xi_t, t \geq 0)$ with state space $E$ starting at $\hat{x} = (s, x) \in E$ by

$$\xi_t := (A_t^{-1} + s, W \circ A_t^{-1}), \qquad \text{for } t \in [0, A_\infty), \tag{2.6}$$

where $W$ is a Brownian motion starting at $x$. Let us denote by $\mathbb{P}_{\hat{x}}^{\xi}$ the law of $\xi$ starting at $\hat{x} \in E$ at time 0. Moreover, we also have to consider the process $\xi$ started at time $t > 0$ and $\hat{x} \in E$ and we denote by $\mathbb{P}_{t,\hat{x}}^{\xi}$ its law, i.e. $\mathbb{P}_{t,\hat{x}}^{\xi}$ is the law of $\xi \circ \theta_{-t}$ under $\mathbb{P}_{\hat{x}}^{\xi}$, where $\xi$ under $\mathbb{P}_{t,\hat{x}}^{\xi}$ is only defined after time $t$. In particular for $s \geq t$,

$$\mathbb{E}_{t,\hat{x}}^{\xi}\big[\varphi(\xi_s)\big] = \mathbb{E}_{\hat{x}}^{\xi}\big[\varphi(\xi_{s-t})\big]. \tag{2.7}$$

Notice that as for every $t \geq 0$, $A_t^{-1}$ is a stopping time with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ we may define the filtration $(\hat{\mathcal{F}}_t, t \geq 0)$ by $\hat{\mathcal{F}}_t := \mathcal{F}(A_t^{-1})$. Then $\xi$ is $\hat{\mathcal{F}}_t$-adapted and we have the following useful lemma, which is proved e.g. in [Dy65a, Chapter X.5].

**Lemma 2.10.** *Let $\hat{T}$ be an $\hat{\mathcal{F}}_t$-stopping time. Then the random variable $T : \Omega \to [0, \infty]$ defined by $T := A_{\hat{T}}^{-1}$ is a $\mathcal{F}_t$-stopping time and $\hat{\mathcal{F}}_{\hat{T}} = \mathcal{F}_T$. Moreover, almost surely, for* *all* $t > 0$,

$$A^{-1}_{\hat{T}+t} = T + A^{-1}_t \circ \theta_T.$$

From the last lemma and the strong Markov property of the Brownian motion $W$ we can deduce that $\xi$ is a strong Markov process with right continuous paths.

**Proposition 2.11 (Strong Markov property of $\xi$).**
*The process $\xi$ is a $E$-valued right continuous strong Markov process with respect to the filtration $(\hat{\mathcal{F}}_t, t \geq 0)$.*

*Proof.* As the paths of $A^{-1}$ are right continuous, the paths of $\xi$ inherit this property in the metric generated by $d_E$. Let $\hat{T}$ be a finite $\hat{\mathcal{F}}_t$-stopping time. By Lemma 2.10, $T := A^{-1}_{\hat{T}}$ is a $\mathcal{F}_t$-stopping time and $\hat{\mathcal{F}}_{\hat{T}} = \mathcal{F}_T$. Hence, for all $\varphi \in \mathcal{B}^b_+(E)$, and all $\hat{x} = (s, x) \in E$, we have $\mathbb{P}^\xi_{\hat{x}}$-almost surely, on the set $\{\hat{T} < A_\infty\}$,

$$\begin{aligned}
\mathbb{E}^\xi_{\hat{x}}\big[\varphi(\xi_{\hat{T}+t}) \,\big|\, \hat{\mathcal{F}}_{\hat{T}}\big] &= \mathbb{E}_x\big[\varphi(A^{-1}_{\hat{T}+t} + s, W \circ A^{-1}_{\hat{T}+t}) \,\big|\, \mathcal{F}_T\big] \\
&= \mathbb{E}_x\big[\varphi(T + s + A^{-1}_t \circ \theta_T, W \circ [T + A^{-1}_t \circ \theta_T]) \,\big|\, \mathcal{F}_T\big] \\
&= \mathbb{E}_{W \circ A^{-1}_{\hat{T}}}\big[\varphi(s + A^{-1}_{\hat{T}} + A^{-1}_t, W \circ A^{-1}_t)\big] \\
&= \mathbb{E}^\xi_{\xi_{\hat{T}}}\big[\varphi(\xi_t)\big],
\end{aligned}$$

where we used Lemma 2.10 for the second and the strong Markov property of the Brownian motion for the third equality. □

**Lemma 2.12 (Continuity in probability of $\xi$).**
*For any $\hat{x} \in E$, the process $\xi$ is continuous in probability under $\mathbb{P}^\xi_{\hat{x}}$.*

*Proof.* Let $\hat{x} \in E$. As the paths $t \mapsto \xi_t$ are right-continuous $\mathbb{P}^\xi_{\hat{x}}$-almost surely, it suffices to show that they also are *left*-continuous in probability on $(0, \infty)$. Let $\eta > 0, \varepsilon > 0$ and $t > 0$. We have to show, that

$$\mathbb{P}^\xi_{\hat{x}}\big(d_E(\xi_{t-\delta}, \xi_t) > \eta\big) < \varepsilon$$

for all $0 < \delta < t$ sufficiently small. As the Brownian motion $W$ is continuous it suffices to show that

$$\mathbb{P}_x\big(|A^{-1}_t - A^{-1}_{t-\delta}| > \eta\big) < \varepsilon$$

for all sufficiently small $\delta > 0$. Applying Lemma 2.10 to the deterministic time $\hat{T} = t - \delta$ we obtain,

$$A^{-1}_t = A^{-1}_{t-\delta} + A^{-1}_\delta \circ \theta_{A^{-1}_{t-\delta}}.$$

Hence, we obtain also using the strong Markov property of the Brownian motion,

$$\mathbb{P}_x\big(|A_t^{-1} - A_{t-\delta}^{-1}| > \eta\big) = \mathbb{P}_x\big(A_\delta^{-1} \circ \theta_{A_{t-\delta}^{-1}} > \eta\big) = \mathbb{E}_x\big[\mathbb{P}_{W \circ A_{t-\delta}^{-1}}\big(A_\delta^{-1} > \eta\big)\big].$$

Also notice that we have as $t \mapsto A_t$ is monotonically increasing,

$$\mathbb{P}_x\big(A_\delta^{-1} > \eta\big) \leq \mathbb{P}\big(A_t \leq \delta \text{ for all } t < \eta\big) = \mathbb{P}_x\big(A_\eta \leq \delta\big).$$

Moreover, as $x \in \mathcal{S}$ we obtain,

$$\lim_{\delta \downarrow 0} \mathbb{P}_x\big(A_\eta \leq \delta\big) = \mathbb{P}_x\Big(\bigcap_{\delta>0} \{A_\eta \leq \delta\}\Big) = \mathbb{P}_x\big(A_\eta = 0\big) = 0,$$

and the proof is complete. □

## 2.2 Statement and proof of the representation theorem

The outline of the section is as follows: Firstly in Section 2.2.1 we introduce a random measure $\Gamma_\sigma$ as the total occupation measure of a suitable (non-catalytic) superprocess and state that this measure is equal in law to the collision local time. As a direct consequence, we obtain the law of its total mass. Section 2.2.2 is then devoted to prepare the proof of the representation theorem: we compute the Laplace functional of $\Gamma_\sigma$. Finally in Section 2.2.3 we prove the representation theorem.

### 2.2.1 The total occupation measure $\Gamma_\sigma$ and the main result

As $\xi$ is a $E$-valued right continuous Markov process we can define a non-catalytic superprocess with spatial motion $\xi$ (see e.g. [LG99, Chapter II]). So, let $\mathcal{M}_f(E)$ be the set of finite measures on $E$ equipped with the weak topology. For $\nu \in \mathcal{M}_f(E)$ and $t \geq 0$, let $\mathbb{P}_{t,\nu}^U$ denote the law of the quadratic (non-catalytic) superprocess $U = (U_s, s \geq t)$ with spatial motion $\xi$, starting at $\nu$ at time $t$. We shall write $\mathbb{P}_\nu^U$ for $\mathbb{P}_{0,\nu}^U$. Recall that $U$ is a time-homogenous $\mathcal{M}_f(E)$-valued branching Markov process. Notice that [LG99, Proposition II.3.8] implies, using Lemma 2.12, that for each $\nu \in \mathcal{M}_f(E)$ the process $s \mapsto U_s$ is continuous in probability under $\mathbb{P}_\nu^U$. Therefore, it is possible to choose a measurable modification of $U$, i.e. that $(s, \omega) \mapsto U_s(\omega)$ is measurable. This measurability allows us to consider integrals of $U$ as in the following definition:

**Definition 2.13 (The total occupation measure).**

*The random measure $\Gamma_\sigma$, defined under $\mathbb{P}^U_{t,\nu}$ by*

$$\Gamma_\sigma(dr,dx) := \int_t^\infty ds\, U_s(dr,dx), \tag{2.8}$$

*is called the* total occupation measure *of the superprocess $U$.*

For $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ let $\nu_\eta$ be the law of the first hitting time and place of $\mathcal{S}$ by a Brownian motion distributed according to $\eta$ at time 0, i.e. define

$$\langle \nu_\eta, \psi \rangle = \int \eta(dx)\, \mathbb{E}_x\big[\mathbf{1}_{\{\tau_\mathcal{S}<\infty\}}\psi(\tau_\mathcal{S}, W_{\tau_\mathcal{S}})\big],$$

where $\tau_\mathcal{S}$ is the first hitting time of $\mathcal{S}$. Notice, as $\eta \in \mathcal{M}_f(\mathbb{R}^d)$ we also have $\nu_\eta \in \mathcal{M}_f(E)$. Let $X = (X_t, t \geq 0)$ be a $\sigma$-catalytic super-Brownian motion and $\mathcal{L}_{\sigma,X}$ the collision local time of $X$ with its catalyst $\sigma$ given by Theorem 1.5. Denote by $\mathbb{P}^X_\eta$ the law of $X$ starting at $X_0 = \eta$. The main result of this chapter is the following theorem:

**Theorem 2.14 (Representation of $\mathcal{L}_{\sigma,X}$).**
*For every $\eta \in \mathcal{M}_f(\mathbb{R}^d)$, the law of the total occupation measure $\Gamma_\sigma$ under $\mathbb{P}^U_{\nu_\eta}$ agrees with the law of the collision local time $\mathcal{L}_{\sigma,X}$ under $\mathbb{P}^X_\eta$ of a $\sigma$-catalytic super-Brownian motion $X$ with its catalytic measure $\sigma$.*

Before we turn our attention to the proof of Theorem 2.14 we draw an easy consequence of the representation theorem: we compute the total mass of the collision local time. Recall, that the 1/2-*stable subordinator* $S = (S_t, t \geq 0)$ is a $\mathbb{R}_+$-valued Lévy process (i.e. a càdlàg process with stationary and independent increments) with non-decreasing paths and starting from zero. Its law $\mathbb{P}^S_0$ can be characterized by its Laplace transform, i.e. for $\lambda \geq 0$,

$$\mathbb{E}^S_0\big[\exp -\lambda S_t\big] = \exp -t\lambda^{1/2}.$$

It is well known, that under $\mathbb{P}^U_\nu$ the *total mass* $\langle \Gamma_\sigma, \mathbf{1}\rangle$ of total occupation measure $\Gamma_\sigma$ of the quadratic superprocess $U$ is distributed as $S_{\langle \nu, 1\rangle}$ under $\mathbb{P}^S_0$. That follows e.g. directly using the Brownian snake representation of superprocesses (see e.g. [LG99]). Moreover,

$$\langle \nu_\eta, \mathbf{1}\rangle = \langle \eta, \mathbb{P}_\cdot(\tau_\mathcal{S} < \infty)\rangle =: \theta_\eta.$$

Hence, we can straightforwardly deduce from Theorem 2.14 the law of the total mass of the collision local time.

**Corollary 2.15 (Distribution of the total mass of $\mathcal{L}_{\sigma,X}$).**
*Let $\eta \in \mathcal{M}_f(\mathbb{R}^d)$. Moreover, denote by $(S_t, t \geq 0)$ a stable subordinator of index 1/2.*

*Then the law of $\langle \mathcal{L}_{\sigma,X}, \mathbf{1}\rangle$ under $\mathbb{P}^X_\eta$ agrees with the law of $S_{\theta_\eta}$ under $\mathbb{P}^S_0$.*

We prove Theorem 2.14 in Section 2.2.3 checking that the characteristic functions of $\Gamma_\sigma$ and $\mathcal{L}_{\sigma,X}$ agree. To prepare this proof, we compute the Laplace functional of the measure $\Gamma_\sigma$ (in Section 2.2.2), also needed later in the thesis.

### 2.2.2 The Laplace functional of $\Gamma_\sigma$

In this section we compute the Laplace functional of $\Gamma_\sigma$ as a special case of the *weighted occupation time formula* see e.g. [Is86, Dy89, LG99]. Let us keep the notation from Section 2.1.

**Lemma 2.16.** *Let $\phi \in \mathcal{B}_+(E)$. The function $w$ defined on $E = \mathbb{R}_+ \times \mathcal{S}$ by*

$$\mathbb{E}^U_\nu\Big[\exp -\langle \Gamma_\sigma, \phi\rangle\Big] = \exp -\langle \nu, w\rangle, \tag{2.10}$$

*is a non-negative solution of the integral equation on $\mathbb{R}_+ \times \mathcal{S}$,*

$$w(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\; w^2(r+s, W_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty dA_r\; \phi(r+s, W_r)\Big]. \tag{2.11}$$

*Proof.* As a special case of the weighted occupation time formula (see e.g. [LG99, Chapter II.3]) we have for all measurable bounded $\phi : E \to \mathbb{R}_+$ and measurable and bounded $h : \mathbb{R}_+ \to \mathbb{R}_+$, such that $h$ has compact support,

$$\mathbb{E}^U_{t,\nu}\Big[\exp - \int_t^\infty ds'\, h(s')\langle U_{s'}, \phi\rangle\Big] = \exp -\langle \nu, w_t\rangle,$$

where $w$ is the unique, non-negative solution of the integral equation

$$w_t(\hat{x}) + \mathbb{E}^\xi_{t,\hat{x}}\Big[\int_t^{A_\infty} ds'\; w^2_{s'}(\xi_{s'})\Big] = \mathbb{E}^\xi_{t,\hat{x}}\Big[\int_t^{A_\infty} ds'\; h(s')\phi(\xi_{s'})\Big].$$

for $t \geq 0$ and $\hat{x} \in E$. By substitution, we have with $\hat{x} = (s,x) \in E$, that

$$w_t(s,x) + \mathbb{E}^\xi_{t,(s,x)}\Big[\int_{A_t^{-1}}^\infty dA_r\; w^2_{A_r}(r, W_r)\Big] = \mathbb{E}^\xi_{t,(s,x)}\Big[\int_{A_t^{-1}}^\infty dA_r\; h(A_r)\phi(r, W_r)\Big].$$

Using the definition of $\mathbb{P}^\xi_{t,\hat{x}}$, this last equation can be rewritten as

$$w_t(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\; w^2_{A_r+t}(r+s, W_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty dA_r\; h(A_r+t)\phi(r+s, W_r)\Big]. \tag{2.12}$$

Using the time homogeneity of the process $U$, we also get that

$$\mathbb{E}^U_{t,\nu}\Big[\exp - \int_t^\infty ds'\, h(s')\langle U_{s'}, \phi\rangle\Big] = \mathbb{E}^U_{\nu}\Big[\exp - \int_0^\infty ds'\, h(s'+t)\langle U_{s'}, \phi\rangle\Big].$$

Using this for $h(t) = \mathbf{1}_{[0,T]}(t)$, we get that the function $w^T$ defined for $t \in [0,T]$ by

$$\mathbb{E}^U_{\nu}\Big[\exp - \int_0^{T-t} ds' \langle U_{s'}, \phi\rangle\Big] = \exp - \langle \nu, w^T_t\rangle,$$

is the unique non-negative solution of (2.12). By monotone convergence, letting $T$ tend to $+\infty$, we get that $w^T_t$ increase pointwise to a function $w$, independent of $t$, defined by (2.10). And $w$ is a non-negative solution of

$$w(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, w^2(r+s, W_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty dA_r\, \phi(r+s, W_r)\Big].$$

Hence the lemma holds for bounded non-negative functions $\phi$. By monotone convergence it also holds for any $\phi \in \mathcal{B}_+(E)$. □

**Remark 2.17.** Note, that according to Lemma 2.16, $w$ is *only* defined on $\mathbb{R}_+ \times \mathcal{S}$. Nevertheless, using the strong Markov property of $W$ at time $\tau_{\mathcal{S}}$, the function $w$ can be extended consistently to $\mathbb{R}_+ \times \mathbb{R}^d$ letting

$$w(s,x) := \mathbb{E}_x\big[\mathbf{1}_{\{\tau_{\mathcal{S}} < \infty\}} w(\tau_{\mathcal{S}} + s, W_{\tau_{\mathcal{S}}})\big].$$

Notice that it is *not* clear that the integral equation (2.11) has a unique solution for *any* $\phi \in \mathcal{B}_+(E)$. However, we show that for *small* test-functions we obtain uniqueness. To prepare the proof of the uniqueness, define the space of test functions $H_b$ by $H_b := \bigcup_{T\geq 0} H_b^T$, where

$$H_b^T := \Big\{\psi \in \mathcal{B}_+^b([0,\infty) \times \mathbb{R}^d) : \operatorname{supp}\psi \subseteq [0,T] \times \mathbb{R}^d\Big\}.$$

**Lemma 2.18.** *Let $\sigma$ be a uniformly non-polar measure on $\mathbb{R}^d$, and $A$ the associated additive functional. Then $a_T := \sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_T] < \infty$ for every $T > 0$.*

*Proof.* Let $T > 0$. Applying (1.5) and Lemma 2.4, we have

$$a_T = \sup_{x\in\mathbb{R}^d} \int_0^T ds \int \sigma(dy)\, p_s(x,y) \leq c \int_0^T ds\, \frac{1}{(s\wedge 1)^{1-\beta}} < \infty$$

for some $c > 0$. □

**Lemma 2.19.** *If $\phi \in H_b^T$ such that $2a_T^2\|\phi\|_\infty < 1$ then (2.11) has a unique non-negative solution.*

*Proof.* Let $\phi \in H_b$ and $w$ be a non-negative solution of (2.11). Then there is a $T > 0$ such that $w(s,x) = 0$ for all $s > T$ and all $x \in \mathcal{S}$. Hence, (2.11) reads,

$$w(s,x) + \mathbb{E}_x\Big[\int_0^T dA_r\, w^2(r+s, W_r)\Big] = \mathbb{E}_x\Big[\int_0^T dA_r\, \phi(r+s, W_r)\Big]. \tag{2.14}$$

Suppose that $\tilde{w} : \mathbb{R}_+ \times \mathcal{S}$ is another non-negative solution of (2.14). Notice that both $w$ and $\tilde{w}$ are bounded by $a_T\,\|\phi\|_\infty$. Hence, we obtain

$$\begin{aligned}\|w - \tilde{w}\|_\infty &\leq \sup_{s\in[0,T],x\in\mathbb{R}^d} \mathbb{E}_x\big[\mathbf{1}_{\{\tau_S<\infty\}}|w^2(\tau_S + s, W_{\tau_S}) - \tilde{w}^2(\tau_S + s, W_{\tau_S})|\big] &(2.15)\\ &\leq \sup_{s\in[0,T],x\in\mathbb{R}^d} \mathbb{E}_x\Big[\int_0^T dA_r\, |w^2(r+s, W_r) - \tilde{w}^2(r+s, W_r)|\Big] \\ &\leq 2a_T^2\|\phi\|_\infty\, \|w - \tilde{w}\|_\infty &(2.16)\end{aligned}$$

which implies $w = \tilde{w}$ as $2a_T^2\|\phi\|_\infty < 1$. □

### 2.2.3 Proof of the representation theorem

Denote by $\mathcal{L}_{\sigma,X}$ the collision local time of a $\sigma$-catalytic super-Brownian motion $X$, starting in $X_0 = \eta$, and its catalytic measure $\sigma$. We prove Theorem 2.14 by checking that the characteristic functions of $\mathcal{L}_{\sigma,X}$ and $\Gamma_\sigma$ agree. For this purpose we need the formulas for the moments of the collision local time which where provided by DELMAS in [De96].

**Lemma 2.20 (Moments of the collision local time).**
*For all $p \geq 1$ and all $\varphi \in \mathcal{B}_+^b(\mathbb{R}_+ \times \mathbb{R}^d)$,*

$$\mathbb{E}_\eta^X\left[\langle \mathcal{L}_{\sigma,X}, \varphi\rangle^p\right] = p! \sum_{k=1}^p \frac{(-1)^{k+p}}{k!} \sum_{n_1+\cdots+n_k=p} \prod_{i=1}^k \langle \eta, c_{n_i}(0,\cdot)\rangle,$$

*where,*

$$\begin{aligned} c_1(s,x) &:= \mathbb{E}_x\Big[\int_0^\infty dA_r\, \varphi(r+s, W_r)\Big] \\ c_n(s,x) &:= -\mathbb{E}_x\Big[\int_0^\infty dA_r \sum_{j=1}^{n-1} c_j(r+s, W_r) c_{n-j}(r+s, W_r)\Big]. \end{aligned}$$

Let us start with the proof of Theorem 2.14. Let $T > 0$, and fix $\varphi \in H_b^T$ such that

$$\|\varphi\|_\infty < \frac{1}{2a_T^2}.$$

First, we show that for all $x \in \mathbb{R}^d$, $s \geq 0$ and all $\lambda \in (0,1)$ the sequence

$$\sum_{n=1}^{\infty} \lambda^n c_n(s,x)$$

is absolutely convergent. We proceed similarly to the proof of Lemma 4.2 in [De96]. The function $f : (0,1) \to \mathbb{R}$ defined by $f(\lambda) := a_T^{-1}[1 - \sqrt{1-\lambda}]$ can be expanded in a power series $f(\lambda) = \sum_{n=0}^{\infty} \beta_n \lambda^n$ for all $\lambda \in (0,1)$. It is easy to check using its definition, that $f$ also satisfies

$$f(\lambda)^2 = \tfrac{1}{a_T}\big(2f(\lambda) - \tfrac{\lambda}{a_T}\big).$$

Moreover, Taylor expansion of $\lambda \mapsto \sqrt{1-\lambda}$ around 0 and comparing the coefficients leads $\beta_0 = 0$ and $\beta_1 = \frac{1}{2a_T}$. By plugging the power series representation for $f$ in the last equation we get, comparing the coefficients, for all $n \geq 2$,

$$\beta_n = \tfrac{1}{2} a_T \sum_{j=1}^{n-1} \beta_{n-j}\beta_j \geq 0.$$

Note that

$$\|c_1\| = \sup_{x \in \mathbb{R}^d, s \geq 0} \mathbb{E}_x\Big[\int_0^\infty dA_r\, \varphi(r+s, W_r)\Big] \leq a_T \|\varphi\|_\infty \leq \tfrac{1}{2a_T} = \beta_1.$$

Moreover, we get by induction on $n$ that for all $n \geq 2$,

$$\|c_n\| \leq a_T \sum_{j=1}^{n-1} \|c_{n-j}\|\,\|c_j\| \leq a_T \sum_{j=1}^{n-1} \beta_{n-j}\,\beta_j = 2\beta_n. \tag{2.20}$$

Hence, the sequence $\sum_{n=1}^{\infty} \lambda^n \|c_n\|$ converges absolutely for all $\lambda \in (0,1)$.

Fix $\lambda \in (0,1)$ for the moment. Expanding the exponential series and using the formulas for the moments, we get

$$\begin{aligned}
\mathbb{E}_\eta^X[\exp -\lambda\langle \mathcal{L}_{\sigma,X},\varphi\rangle] &= \sum_{p=0}^{\infty} \tfrac{(-\lambda)^p}{p!}\,\mathbb{E}_\eta^X\big[\langle \mathcal{L}_{\sigma,X},\varphi\rangle^p\big] \\
&= 1+\sum_{p=1}^{\infty}\sum_{k=1}^{p}\lambda^p\,\tfrac{(-1)^k}{k!}\sum_{n_1+\cdots+n_k=p}\prod_{i=1}^{k}\langle\eta, c_{n_i}(0,\cdot)\rangle \\
&= \sum_{k=0}^{\infty}\tfrac{(-1)^k}{k!}\Big[\sum_{n=1}^{\infty}\langle\eta,\lambda^n c_n(0,\cdot)\rangle\Big]^k \\
&= \exp -\Big\langle \eta, \sum_{n=1}^{\infty}\lambda^n c_n(0,\cdot)\Big\rangle.
\end{aligned}$$

By the Lemmas 2.16 and 2.19 the function $w_\lambda : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ defined by

$$\mathbb{E}_{\nu_\eta}^U\big[\exp -\lambda\langle \Gamma_\sigma,\varphi\rangle\big] = \exp -\langle \nu_\eta, w_\lambda\rangle, \tag{2.21}$$

is the unique non-negative solution of

$$w_\lambda(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\; w_\lambda^2(r+s,W_r)\Big] = \lambda\mathbb{E}_x\Big[\int_0^\infty dA_r\;\varphi(r+s,W_r)\Big]. \tag{2.22}$$

Also recall, that $w_\lambda$, can be extended consistently to $\mathbb{R}_+\times\mathbb{R}^d$ letting

$$w_\lambda(s,x) := \mathbb{E}_x\big[\mathbf{1}_{\{\tau_S<\infty\}} w_\lambda(\tau_S+s, W_{\tau_S}\big].$$

The first step to complete the proof is to show that the function $\tilde{w}_\lambda : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}_+$ defined by

$$\tilde{w}_\lambda(x,s) := \sum_{n=1}^{\infty}\lambda^n c_n(s,x)$$

solves (2.22). Notice that $\tilde{w}$ is well-defined as the sequence $\sum_{n=1}^{\infty}\lambda^n\|c_n\|$ converges absolutely for all $\lambda\in(0,1)$.

By definition of $c_1$, we have

$$c_1(s,x) = \mathbb{E}_x\Big[\int_0^\infty dA_r\,\varphi(r+s,W_r)\Big]. \tag{2.23}$$

Moreover, as $\varphi \in H_b^T$,

$$
\begin{aligned}
-\mathbb{E}_x\Big[\int_0^\infty & dA_r\,\tilde{w}_\lambda^2(r+s,W_r)\Big] \\
&= -\mathbb{E}_x\Big[\int_0^\infty dA_r\,\big(\sum_{n=1}^\infty \lambda^n c_n(r+s,W_r)\big)^2\Big] \\
&= -\mathbb{E}_x\Big[\int_0^\infty dA_r\,\sum_{n=1}^\infty\sum_{j=1}^n \big(\lambda^{n+1-j}\,c_{n+1-j}(r+s,W_r)\,\lambda^j\,c_j(r+s,W_r)\big)\Big] \\
&= -\mathbb{E}_x\Big[\int_0^\infty dA_r\,\mathbf{1}_{\{r<T\}}\sum_{n=1}^\infty\sum_{j=1}^n \big(\lambda^{n+1}\,c_{n+1-j}(r+s,W_r)\,c_j(r+s,W_r)\big)\Big] \\
&= \sum_{n=1}^\infty -\lambda^{n+1}\mathbb{E}_x\Big[\int_0^\infty dA_r\,\mathbf{1}_{\{r<T\}}\sum_{j=1}^n c_{n+1-j}(r+s,W_r)\,c_j(r+s,W_r)\Big] \\
&= \sum_{n=1}^\infty \lambda^{n+1}c_{n+1}(s,x) = \sum_{n=2}^\infty \lambda^n c_n(s,x). \qquad (2.24)
\end{aligned}
$$

Note, that we use bounded convergence for the fourth equality, which is allowed since, by (2.20), the modulus of the double sum in the third line is bounded by

$$(2/a_T)\sum_{n=1}^\infty \beta_n\lambda^n < \infty.$$

Hence, combining (2.23) and (2.24), we see that $\tilde{w}_\lambda$ solves the integral equation (2.22). As $w_\lambda$ is non-negative and $\varphi \in H_b^T$, it is easy to see from (2.22) that $w_\lambda(s,x) = 0$ for all $s \geq T$. Moreover, $c_n(s,x) = 0$ for all $s \geq T$, and hence we also have $\tilde{w}_\lambda(s,x) = 0$ for all $s \geq T$. Thus, as both $w_\lambda$ and $\tilde{w}_\lambda$ are bounded by $a_T\,\|\varphi\|_\infty$, we can estimate,

$$
\begin{aligned}
\|w_\lambda - \tilde{w}_\lambda\|_\infty &\leq \sup_{s\in[0,T],x\in\mathbb{R}^d} \mathbb{E}_x\big[\mathbf{1}_{\tau_S<\infty}\big|w_\lambda(\tau_S+s,W_{\tau_S}) - \tilde{w}_\lambda(\tau_S+s,W_{\tau_S})\big|\big] \\
&\leq \sup_{s\in[0,T],x\in\mathbb{R}^d} \mathbb{E}_x\Big[\int_0^T dA_r\,|w_\lambda^2(r+s,W_r) - \tilde{w}_\lambda^2(r+s,W_r)|\Big] \\
&\leq 2a_T^2\|\varphi\|_\infty\,\|w_\lambda - \tilde{w}_\lambda\|_\infty,
\end{aligned}
$$

which implies $\tilde{w}_\lambda = w_\lambda$ on $\mathbb{R}_+ \times \mathbb{R}^d$, as $2a_T^2\,\|\varphi\|_\infty < 1$.

Hence, using the definition of $\nu_\eta$,

$$\langle \nu_\eta, w_\lambda\rangle = \int \eta(dy)\,\mathbb{E}_y\big[\mathbf{1}_{\{\tau_S<\infty\}}w_\lambda(\tau_S,W_{\tau_S})\big] = \langle \eta, w_\lambda(0,\cdot)\rangle.$$

Let $\varphi \in H_b$. Then $\varphi \in H_b^T$ for some $T > 0$ and we can choose $\lambda > 0$ so small such that $2a_T^2 \|\lambda\varphi\|_\infty < 1$. Therefore, for *every* $\varphi \in H_b$, and all sufficiently small $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}_\eta^X \left[ \exp -\lambda \langle \mathcal{L}_{\sigma,X}, \varphi \rangle \right] &= \exp -\langle \eta, \tilde{w}_\lambda(0, \cdot) \rangle \\
&= \exp -\langle \nu_\eta, w_\lambda \rangle \\
&= \mathbb{E}_{\nu_\eta}^U \left[ \exp -\lambda \langle \Gamma_\sigma, \varphi \rangle \right],
\end{aligned}$$

hence all moments of $\langle \mathcal{L}_{\sigma,X}, \varphi \rangle$ and $\langle \Gamma_\sigma, \varphi \rangle$ agree. The power series built from these moments has a positive radius of convergence, and by analytic continuation the characteristic functions also agree. Hence both random measures have the same distribution which completes the proof of the theorem. □

# Chapter 3

# Catalytic super-Brownian motion via collision local time

*(This chapter is based on a joint work[1] with Peter Mörters, Bath.)*

In this chapter we show how to construct a class of $\sigma$-catalytic super-Brownian motions deterministically from the random measure $\Gamma_\sigma$. Our result gives an independent construction of catalytic super-Brownian motion for a large class of catalysts.

Similar methods are then applied to construct a catalytic superprocess in a bounded domain with reflecting boundary which is catalyzed by the surface measure on a part of its boundary. As underlying motion of this superprocess serves a *reflecting Brownian motion* in this domain. Therefore, we shall speak of super-Brownian motion with reflecting boundary.

The chapter is organized as follows: Section 3.1 reviews the main technical tool for this chapter: *excursions theory and exit systems* and applies the general theory to obtain the *last exit decomposition* of standard Brownian motion. In Section 3.2 we prove Theorem 1.7: the construction of $\sigma$-catalytic super-Brownian motion. Then, Section 3.3 reviews some properties and excursion theory for reflecting Brownian motion in a bounded domain $D$ and provides all we need to construct the catalytic super-Brownian motion in $D$ in Section 3.4

[1] See also [MV03]; submitted to *Stochastic Processes and their Applications.*

## 3.1 Excursion theory and Maisonneuve's exit systems

The main technical tool for the construction of catalytic super-Brownian motion in $\mathbb{R}^d$ or in a bounded domain $D$ with reflecting boundary is the theory of excursions from *subsets* of the state-space. In particular, we need the excursion theory from the catalytic set, i.e. the support of the catalytic measure, for standard and reflecting Brownian motion respectively. In this section, we present some results of the pioneering work done by MAISONNEUVE, who introduced the so called *exit systems* in a very general setting. Indeed, he considered arbitrary Markov processes taking values in a metric space $E$ (see [Ma75]).

For our purpose, we restrict ourselves to the case of continuous strong Markov processes taking values in $E \subseteq \mathbb{R}^d$. Of course, this contains both cases we are interested in: the standard *and* the reflecting Brownian motion. Firstly, in Section 3.1.1 we show, for a fixed subset $C$ of the state space, how to construct a natural continuous additive functional of the underlying Markov process, the so called *capacitary local time* on $C$. Moreover, we justify this definition characterizing its Revuz-measure as the *capacitary measure* of the set $C$. Then, following [Ma75], Section 3.1.2 reviews the general excursion theory. Finally, in Section 3.1.3 we apply the general results to obtain the *last exit decomposition* of standard Brownian motion.

### 3.1.1 Capacitary local time

Let $Y = (Y_t, t \geq 0)$ be a continuous strong Markov process taking values in $E \subseteq \mathbb{R}^d$. Denote by $\mathbb{P}_x^Y$ the law of $Y$ starting in $Y_0 = x \in E$ and let $(\mathcal{F}_t, t \geq 0)$ be the right continuous filtration generated by $Y$ completed in the usual way. Suppose that $Y$ is symmetric in the sense that it has a transition density $p_t(x, y)$ which is symmetric in $x$ and $y$. Moreover, let $C \subseteq E$ be regular for itself and non-polar for $Y$. We denote by $\tau_C$ the first hitting time of $C$ by the Markov process $Y$, i.e.

$$\tau_C := \inf\{t > 0 : Y_t \in C\}.$$

Notice that $\tau_C$ is an $\mathcal{F}_t$-stopping time. For any set $A \subseteq \mathbb{R}^d$ denote by $\overline{A}$ the closure of $A$ and set

$$M := \overline{\{t > 0 : Y_t \in C\}},$$

and notice that the set $M$ is time homogeneous, i.e. for all $t \geq 0$,

$$(M - t) \cap (0, \infty) = \overline{\{s > 0 : Y_s \circ \theta_t \in C\}} = M \circ \theta_t,$$

where $(\theta_t, t \geq 0)$ is the family of shift operators introduced in Section 2.1.1. We want to construct a *natural* continuous additive functional of $Y$ which grows exactly on the set $M$. This is done is the following Proposition. Recall the definition of a regular potential from Section 2.1.2.

**Proposition 3.1 (Existence and definition of the capacitary local time $L$).** *The function $\psi(x) := \mathbb{E}_x[\mathrm{e}^{-\tau_C}]$ on $C$ is a regular 1-potential. In particular, there is a unique continuous additive functional $L = (L_t, t \geq 0)$ of $Y$ called the* capacitary local time *on $C$ such that*

$$\mathbb{E}_x^Y\Big[\int_0^\infty \mathrm{e}^{-t}\, dL_t\Big] = \mathbb{E}_x^Y\big[\mathrm{e}^{-\tau_C}\big].$$

*Proof.* For a proof of the first statement we refer to [Bl92, Chapter VII.2]. The 'in particular statement' then follows using the Volkonskii-Sŭr-Meyer Theorem 2.3. □

Moreover, we have that $\operatorname{supp} L = C$ and $L$ only grows on the set $M$ (also see [Bl92, Chapter VII.2]), in fact we have almost surely for all $t \geq 0$,

$$L_t = \int_0^t dL_s\, \mathbf{1}_C(Y_s)$$

and therefore $L$ is a local time on $C$. In the following, we like to justify the particular name of *capacitary* local time.

**Definition 3.2 (Capacitary measure).** *The measure $\kappa$ defined on $E$ such that for any $\psi \in \mathcal{B}_+^b(E)$ we have,*

$$\langle \kappa, \psi \rangle = \int_{\mathbb{R}^d} dx\, \mathbb{E}_x^Y\big[\mathbf{1}_{\{\tau_C < \infty\}}\, \mathrm{e}^{-\tau_C}\, \psi(Y_{\tau_C})\big],$$

*is called the* capacitary measure *of $C$.*

This definition is the special case $(q = 1)$ of the usual definition (see e.g. [Be96, Chapter II]) where the $q$-capacitary measure is defined. In other words, the capacitary measure $\mu$ of the set $C$ is the distribution of the first point visited in $C$ by strong Markov process $Y$, where $Y$ is killed at rate 1 and originally distributed according to the Lebesgue measure. Moreover, one can show that $\kappa$ is a Radon measure whose closed support is contained in $C$ (see e.g. [Be96, Chapter II]).

Let $\mu$ be the Revuz-measure associated to the continuous additive functional $L$ by

$$\mathbb{E}_x^Y\Big[\int_0^\infty dL_t\, \varphi(t, Y_t)\Big] = \int \mu(dy) \int_0^\infty dt\, \varphi(t, y) p_t(x, y).$$

**Proposition 3.3 (Characterization of the measure $\mu$).**
*The Revuz measure $\mu$ associated to $L$ agrees with the capacitary measure of $C$.*

*Proof.* Let $\kappa$ be the capacitary measure of $C$, i.e. $\kappa$ is defined by $\kappa(dy) = \int_{\mathbb{R}^d} dz\, h(z, dy)$, where for any $x \in \mathbb{R}^d$, the measure $h(x, dy)$ on $C$ is defined by

$$h(x, A) = \mathbb{E}_x^Y\left[\mathbf{1}_{\{\tau_C < \infty\}}\, \mathrm{e}^{-\tau_C}\, \mathbf{1}_A(Y_{\tau_C})\right],$$

for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$. Moreover, let $\mu$ be the Revuz-measure of the additive functional $L$. We want to show that $\kappa = \mu$. Recall, that we denote by $p_t(x, y)$ the transition density of $Y$ and that the 1-potential $u_\nu^1 \in \mathcal{B}_+(\mathbb{R}^d)$ of a measure $\nu$ on $\mathbb{R}^d$ is defined by

$$u_\nu^1(x) := \int \nu(dy) \int_0^\infty dt\ \mathrm{e}^{-t}\, p_t(x, y),$$

Notice that as

$$u_\mu^1(x) = \int \mu(dy) \int_0^\infty dt\ \mathrm{e}^{-t}\, p_t(x, y) = \mathbb{E}_x^Y[\mathrm{e}^{-\tau_C}] \leq 1,$$

the 1-potential of $\mu$ is bounded and it is enough to check (see e.g. [BlG68, Prop. VI.1.15]) that $u_\mu^1 = u_\kappa^1$ almost everywhere on $\mathbb{R}^d$ in order to deduce $\mu = \kappa$. Let $\psi$ be a non-negative bounded measurable function defined on $\mathbb{R}^d$. We have,

$$\begin{aligned}
\int_{\mathbb{R}^d} dx\, u_\kappa^1(x)\psi(x) &= \int_{\mathbb{R}^d} dx\, \psi(x) \int \kappa(dy) \int_0^\infty dt\ \mathrm{e}^{-t}\, p_t(x, y) \\
&= \int_{\mathbb{R}^d} dx\, \psi(x) \int_0^\infty dt\ \mathrm{e}^{-t} \int_{\mathbb{R}^d} dz \int p_t(x, y) h(z, dy) \\
&= \int_{\mathbb{R}^d} dx\, \psi(x) \int_0^\infty dt\ \mathrm{e}^{-t} \int_{\mathbb{R}^d} dz \int p_t(y, x) h(z, dy) \\
&= \int_{\mathbb{R}^d} dz\ \mathbb{E}_z^Y\Big[\mathrm{e}^{-\tau_C}\, \mathbb{E}_{Y_{\tau_C}}^Y\Big[\int_0^\infty dt\ \mathrm{e}^{-t}\, \psi(Y_t)\Big]\Big] \\
&= \int_{\mathbb{R}^d} dz\ \mathbb{E}_z^Y\Big[\int_{\tau_C}^\infty dt\ \mathrm{e}^{-t}\, \psi(Y_t)\Big] \\
&= \int_{\mathbb{R}^d} dz\ \mathbb{E}_z^Y\Big[\int_0^\infty dt\ \mathrm{e}^{-t}\, \psi(Y_t)\Big] - \int_{\mathbb{R}^d} dz\ \mathbb{E}_z^Y\Big[\int_0^{\tau_C} dt\ \mathrm{e}^{-t}\, \psi(Y_t)\Big],
\end{aligned}$$

where we used the symmetry of $p$ for the third and the strong Markov property for the fifth equality. Now, using again the symmetry of $p$ for the first term of the last equation, we get

$$\begin{aligned}
\int_{\mathbb{R}^d} dz\, \mathbb{E}_z^Y\Big[\int_0^\infty dt\ \mathrm{e}^{-t}\,\psi(Y_t)\Big] &= \int_{\mathbb{R}^d} dz \int_0^\infty dt\ \mathrm{e}^{-t} \int_{\mathbb{R}^d} dy\ p_t(z,y)\psi(y) \\
&= \int_{\mathbb{R}^d} dz \int_0^\infty dt\ \mathrm{e}^{-t} \int_{\mathbb{R}^d} dy\ p_t(y,z)\psi(y) \\
&= \int_{\mathbb{R}^d} dy\ \psi(y).
\end{aligned}$$

Let $p_t^C$ be the density of the transition kernel of $Y$ killed on $C$. For $t > 0$, the function $p_t^C(x,y)$ is symmetric (see e.g. [CZ95, Chapter II]). For the second term, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} dz\, \mathbb{E}_z^Y\Big[\int_0^{\tau_C} dt\ \mathrm{e}^{-t}\,\psi(Y_t)\Big] &= \int_{\mathbb{R}^d} dz \int_0^\infty dt\ \mathrm{e}^{-t} \int_{\mathbb{R}^d} dy\ p_t^C(z,y)\psi(y) \\
&= \int_{\mathbb{R}^d} dz \int_0^\infty dt\ \mathrm{e}^{-t} \int_{\mathbb{R}^d} dy\ p_t^C(y,z)\psi(y) \\
&= \int_{\mathbb{R}^d} dy\ \psi(y)\mathbb{E}_y^Y\Big[\int_0^{\tau_C} dt\ \mathrm{e}^{-t}\Big].
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\int_{\mathbb{R}^d} dx\, u_\kappa^1(x)\psi(x) &= \int_{\mathbb{R}^d} dy\ \psi(y) - \int_{\mathbb{R}^d} dy\ \psi(y)\mathbb{E}_y^Y\Big[\int_0^{\tau_C} dt\ \mathrm{e}^{-t}\Big] \\
&= \int_{\mathbb{R}^d} dy\ \psi(y)\mathbb{E}_y^Y[\mathrm{e}^{-\tau_C}] \\
&= \int_{\mathbb{R}^d} dy\ \psi(y)\mathbb{E}_y^Y\Big[\int_0^\infty dL_t\ \mathrm{e}^{-t}\Big] \\
&= \int_{\mathbb{R}^d} dy\ \psi(y) \int_{\mathbb{R}^d} \mu(dz) \int_0^\infty dt\ \mathrm{e}^{-t} p_t(y,z) \\
&= \int_{\mathbb{R}^d} dy\, u_\mu^1(y)\psi(y)\ dy.
\end{aligned}$$

And we get $u_\mu^1 = u_\kappa^1$ almost everywhere in $\mathbb{R}^d$. Thus we have $\mu = \kappa$. □

### 3.1.2 A general excursion formula

In order to formulate the general excursion theorem, we have to introduce some notation. Recall from the last section $M = \overline{\{t > 0 : Y_t \in C\}}$. Following [Ma75], we set

$$\begin{aligned}
R &:= \inf\{s > 0\ :\ s \in M\}, \\
R_t &:= \inf\{s > 0\ :\ s + t \in M\} = R \circ \theta_t.
\end{aligned}$$

In fact, $R = \tau_C$ is the first hitting time of $C$ and $R_t$ is the first hitting time of $C$ after time $t$. The paths $t \mapsto R_t$ are càdlàg (right continuous with left limits) and we denote by $R_{t-} := \lim_{s \uparrow t} R_s$. Then, $\{t > 0 : R_{t-} = 0\} = M$, and define the set

$$G := \{t > 0 : R_{t-} = 0, R_t > 0\}.$$

The set $G$, is the set of *left endpoints* in $(0, \infty)$ of the intervals contiguous to $M$. Notice, $G$ is countable and $G \subset M$ almost surely. Let $\delta$ be a cemetery point added to $E$ and consider a metric on $E^\delta := E \cup \{\delta\}$ under which $E^\delta$ is a compact metric space. Moreover, let $\mathcal{D} = \mathcal{D}(\mathbb{R}_+, E^\delta)$ be the Skorokhod space of càdlàg functions defined on $\mathbb{R}_+$ endowed with the Skorokhod topology. Notice that the Skorokhod topology is a metrizable topology under which $\mathcal{D}$ becomes a complete separable metric space. Actually, the extra point $\delta$ is only needed for the definition of the excursions in MAISONNEUVES setting. We denote by $\omega_\delta$ the function constantly equal to $\delta$ on $\mathbb{R}_+$. If $\omega \in \mathcal{D}$, we write indifferently $\omega(t)$ or $\omega_t$ to denote the value of the function $\omega$ at time $t$. For $s > 0$, let $i_s : \mathcal{D} \to \mathcal{D}$ be the family of translation operators defined by,

$$\begin{aligned} &i_s(\omega)(t) = \omega(t+s) \quad \text{for } 0 \leq t < R_s, \\ &i_s(\omega)(t) = \delta \quad \text{for } t \geq R_s. \end{aligned}$$

**Definition 3.4 (Excursions).**
*The collection $\{i_s(\omega) : s \in G(\omega)\}$ is called the collection of* excursions *with respect to $M$ of the path $\omega$. Moreover, $\zeta(i_s(\omega)) := \inf\{t > 0 : i_s(\omega)(t) = \delta\}$ is called the* length *of the excursion $i_s(\omega)$.*

Roughly speaking an excursion of the process $Y$ is a part of its path $\{Y_t, s \leq t < R_s\}$ starting in some point $x \in C$ at time $s \in G$ up to the first time it hits $C$ again. In the sequel we shall just write $\omega$ to denote an excursion as well as a whole path if there is no ambiguity.

Recall that we denote by $(\mathcal{F}_t, t \geq 0)$ the filtration generated by $Y$ and let $\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t$ the sigma-field generated by $Y$. Moreover, let $(Q_t, t \geq 0)$ be the transition kernels of the Markov process $Y$ killed on the set $C$, i.e.

$$Q_t \varphi(x) = \mathbb{E}_x^Y \left[ \mathbf{1}_{\{t < \tau_C\}} \varphi(Y_t) \right].$$

Recall that loosely speaking the previsible processes are those obtained by linear and monotone operations from processes of the form $s \mapsto \mathbf{1}_{[0,R]}(s)$, where $R$ is an $\mathcal{F}_t$-stopping time. To be rigorous, let $\mathcal{P}$ be the sigma-field generated by all $\mathcal{F}_t$-adapted left continuous processes. Then an $\mathcal{F}_t$-adapted process $Z = (Z_s, s \geq 0)$ is called *previsible* if it is measurable with respect to $\mathcal{P}$. Still following [Ma75] we recall the main tool of excursion theory from sets:

**Theorem 3.5 (Maisonneuve's exit formula).**
*There exists a family of sigma-finite measures $H = (H^x, x \in C)$ on $(\mathcal{D}, \mathcal{F})$, such that for any positive, previsible process $Z = (Z_s, s \geq 0)$ and for any $\mathcal{F}$-measurable non-negative function $\psi$ on $\mathcal{D}$, such that $\psi(w_\delta) = 0$, we have*

$$\mathbb{E}_x^Y\Big[\sum_{s \in G} Z_s \psi \circ i_s\Big] = \mathbb{E}_x^Y\Big[\int_0^\infty Z_s H^{Y_s}[\psi] dL_s\Big].$$

*Furthermore, the measure $H^x$ is strong Markov with respect to the semigroup $(Q_t, t \geq 0)$, i.e. for every $\mathcal{F}_t$-stopping time $T$ we have for all $\mathcal{F}_T$ measurable $\varphi \geq 0$,*

$$H^x\big[\varphi \cdot \psi \circ i_T\big] = H^x\big[\varphi \cdot Q_T \psi\big].$$

**Definition 3.6 (Excursion measures and exit system).**
*The family of measures $H$ is called the family of* excursion measures *of the set $C$. The pair $(H, L)$ is called is the* exit system *associated with $C$.*

Using a monotone class argument (simply use $\mathbf{1}_{[0,T]}(s) Z_s$ instead of $Z_s$ for deterministic $T > 0$), Theorem 3.5 implies the following corollary:

**Corollary 3.7.** *For all positive, previsible processes $Z = (Z_s, s \geq 0)$ and for any non-negative function $\psi \in \mathcal{B}_+(\mathbb{R}) \otimes \mathcal{F}$ such that $\psi(\cdot, \omega_\delta) = 0$ we have,*

$$\mathbb{E}_x^Y\Big[\sum_{s \in G} Z_s \psi(s, \cdot) \circ i_s\Big] = \mathbb{E}_x^Y\Big[\int_0^\infty Z_s H^{Y_s}[\psi(s, \cdot)] dL_s\Big].$$

Let us consider the following important example: Let $\phi$ be a non-negative $\mathcal{F}$-measurable function on $\mathcal{D}$ such that $\phi(\omega_\delta) = 0$ and let $t > 0$. Define $\psi : \mathbb{R}_+ \times \mathcal{D} \to \mathbb{R}_+$ by

$$\psi(s, \omega) := \phi \circ i_{t-s}(\omega) \mathbf{1}_{[0,t)}(s).$$

Hence, $\psi \in \mathcal{B}_+(\mathbb{R}_+) \otimes \mathcal{F}$ and we deduce

$$\begin{aligned}\mathbb{E}_x^Y\Big[\sum_{s \in G} Z_s \psi(s, \cdot) \circ i_s\Big] &= \mathbb{E}_x^Y\Big[\sum_{s \in G} Z_s \mathbf{1}_{\{s<t\}} \phi \circ i_{t-s} \circ i_s\Big] \\ &= \mathbb{E}_x^Y\big[\mathbf{1}_{\{R<t, t \notin M\}} Z_{g_t} \phi \circ i_t\big],\end{aligned}$$

where $g_t := \sup\{s < t : s \in M\}$. Taking $Z := 1$ and $\phi(\omega) := \varphi(\omega_0)$ for $\varphi \in \mathcal{B}_+(E)$ if

$\omega \neq \omega_\delta$ and $\phi(\omega) = 0$ if $\omega = \omega_\delta$, we obtain using Corollary 3.7 and writing $C^c := E \setminus C$,

$$\mathbb{E}_x^Y\Big[\int_0^\infty dL_s\, \mathbf{1}_{\{s<t\}} H^{Y_s}[\varphi(\omega_{t-s})]\Big] = \mathbb{E}_x^Y\big[\varphi(Y_t)\mathbf{1}_{\{t>\tau_C\}}\mathbf{1}_{C^c}(Y_t)\big]. \tag{3.1}$$

In particular, we need equation (3.1) for the special cases when $Y$ is either a standard Brownian motion in $E = \mathbb{R}^d$ or a reflecting Brownian motion in a bounded domain $E = D \subseteq \mathbb{R}^d$. The first case is treated in the next section.

### 3.1.3 Last exit decomposition for standard Brownian motion

In this section we consider $Y$ to be a standard Brownian motion $W = (W_t, t \geq 0)$ in $\mathbb{R}^d$. Recall, that we write $\mathbb{P}_x$ for the law of $W$ starting in $x \in \mathbb{R}^d$. Moreover, let $\sigma$ be a uniformly non-polar measure on $\mathbb{R}^d$ and $\mathcal{S}$ be the fine support of $\sigma$. We also assume that $C = \mathcal{S}$ is a set of *vanishing* $d$-dimensional Lebesgue measure. According to Section 3.1.2, there is a continuous additive functional $L$ of $W$, called the capacitary local time on $\mathcal{S}$, such that,

$$\mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-t}\, dL_t\Big] = \mathbb{E}_x\big[\mathrm{e}^{-\tau_S}\big],$$

where $\tau_S$ denotes the first hitting time of $\mathcal{S}$ by the Brownian motion $W$. If $\mathcal{S}$ has vanishing $d$-dimensional Lebesgue measure we obtain from (3.1)

$$\mathbb{E}_x[\varphi(W_t)] = Q_t\varphi(x) + \mathbb{E}_x\Big[\int_0^\infty \mathbf{1}_{\{s<t\}} H^{W_s}[\varphi(\omega_{t-s})]\, dL_s\Big], \tag{3.2}$$

where $(Q_t, t \geq 0)$ denotes the semigroup of Brownian motion killed on $\mathcal{S}$. If we denote by $\zeta(\omega)$ the length of the excursion $\omega$, then we let $\varphi(\omega_t) = 0$ for all $t \geq \zeta(\omega)$ on the right hand side of (3.2). The family of excursion measures $(H^x, x \in \mathcal{S})$ satisfies the *entrance law equation* for the semigroup $(Q_t, t \geq 0)$ of Brownian motion killed on $\mathcal{S}$, see e.g. [Bl92, Chapter V], i.e., for all $t \geq 0, s > 0$,

$$H^x[\varphi(\omega_{t+s})] = \int H^x[\omega_s \in dy] \int \varphi(z)\, Q_t(y, dz). \tag{3.3}$$

From (3.3) and the fact that $Q_t(y, \cdot)$ has a density one can see easily that each $H^x[\omega_s \in dy]$ has a density on $\mathcal{S}^c$.

Intuitively, equation (3.2) separates the computation of the transition semigroup of Brownian motion in two parts: the first arises from paths that do *not* hit $\mathcal{S}$ and the second from excursions starting at the *last-exit-time* from $\mathcal{S}$ which is distributed according to the measure $dL$. Therefore, we refer to equation (3.2) to be the *last-exit-decomposition* of standard Brownian motion.

## 3.2 $\sigma$-catalytic super-Brownian motion in $\mathbb{R}^d$

This section is devoted to construct the $\sigma$-catalytic super-Brownian motion from the collision local time. The section is organized as follows: first we restate the main result Theorem 1.7. Then in the Sections 3.2.1 and Section 3.2.2 we give the proof of Theorem 1.7 part (a) and (b) respectively. Moreover in Section 3.2.3 we deduce from our construction that the catalytic super-Brownian motion possesses a smooth density which solves the heat equation outside the catalyst with (random) boundary condition given by the collision local time. Finally, Section 3.2.4 is devoted to enlighten the abstract absolute continuity condition needed for Theorem 1.7.

Let $\sigma$ be a uniformly non-polar measure on $\mathbb{R}^d$, $\mathcal{S}$ its fine support, i.e. the support of the associated continuous additive functional $A$. Moreover, denote by $L$ the capacitary local time of a $d$-dimensional Brownian motion $W$ on the set $\mathcal{S}$. Recall, that the Revuz-measure $\mu$ of $L$ is the capacitary measure of the set $\mathcal{S}$. Let us restate the result we are going to prove:

**Theorem 1.7 (Construction of $\sigma$-catalytic super-Brownian motion).** *Assume that $\sigma$ is a uniformly non-polar measure such that its fine support $\mathcal{S}$ has vanishing Lebesgue measure. Moreover, let the capacitary measure $\mu$ of $\mathcal{S}$ be absolutely continuous with respect to $\sigma$, i.e. there is a density $f$ such that $\mu(dx) = f(x)\sigma(dx)$.*

*(a) The measure valued process $Z = (Z_t, t \geq 0)$ defined by $Z_0 := \eta \in \mathcal{M}_f(\mathbb{R}^d)$ and*

$$\langle Z_t, \varphi\rangle = \langle \eta, Q_t\varphi\rangle + \iint \Gamma_\sigma(ds\, dx)\, \mathbf{1}_{\{s<t\}} f(x)\, H^x[\varphi(\omega_{t-s})], \tag{3.4}$$

*for any $\varphi \in \mathcal{B}^b_+(\mathbb{R}^d)$ and all $t > 0$, is a $\sigma$-catalytic super-Brownian motion $X$ with start in $X_0 = \eta$.*

*(b) The collision local time $\mathcal{L}_{\sigma,Z}$ of $Z$ with the catalytic measure $\sigma$ is given by the random measure $\Gamma_\sigma$.*

Although we showed in Chapter 2 that the collision local time $\mathcal{L}_{\sigma,X}$ of a $\sigma$-catalytic super-Brownian motion $X$ with its catalyst $\sigma$ is *distributed* as the random measure $\Gamma_\sigma$, it is not clear that the collision local time of the process $Z$ defined by (3.4) is given by $\Gamma_\sigma$ — even if we have already shown that $Z$ is a $\sigma$-catalytic super-Brownian motion.

According to the definition of the collision local time (see Theorem 1.5) we have to show that for all $\varphi \in H_b$, almost surely

$$\lim_{\varepsilon\downarrow 0}\int_0^\infty dt \int \sigma(dy)\int Z_t(dx)\, p_\varepsilon(x,y)\psi(t,y) = \iint \Gamma_\sigma(dr\, dx)\,\psi(r,x).$$

But before we prove this, we show part (a) of the theorem.

### 3.2.1 Proof of Theorem 1.7 (a)

We start by stating and proving a preliminary lemma on catalytic super-Brownian motion.

**Lemma 3.8.** *Let $X$ be a $\sigma$-catalytic super-Brownian motion with start in $X_0 = \eta$. Then the finite dimensional marginals of $X$ can be characterized in the following way: For all $0 < t_1 < t_2 < \cdots < t_n < T$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{B}_+^b(\mathbb{R}^n)$ such that $2a_T \sum_{i=1}^n \|\varphi_i\|_\infty < 1$ we have*

$$\mathbb{E}_\eta^X\Big[\exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle\Big] = \exp - \langle \eta, u(0,\cdot)\rangle,$$

*where $u$ is the unique non-negative solution of*

$$u(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, u^2(r+s, W_r)\Big] = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\, \mathbb{E}_x\big[\varphi_i(W_{t_i - s})\big]. \tag{3.5}$$

*Proof.* We prove the lemma by induction on $n$ essentially following [LG99, Chapter II]. Let $n = 1$. Let $v$ be a solution of (1.7) with $\varphi = \varphi_1$, then

$$u(s,x) = \mathbf{1}_{\{s\le t_1\}} v(t_1 - s, x)$$

solves (3.5) and

$$\mathbb{E}_\eta^X\big[\exp - \langle X_{t_1}, \varphi_1\rangle\big] = \exp - \langle \eta, v(t_1,\cdot)\rangle = \exp - \langle \eta, u(0,\cdot)\rangle.$$

Let $n \ge 2$ and assume that the result holds up to order $n-1$. By the Markov property at time $t_1$,

$$\begin{aligned}\mathbb{E}_\eta^X\Big[\exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle\Big] &= \mathbb{E}_\eta^X\Big[\exp - \langle X_{t_1}, \varphi_1\rangle\, \mathbb{E}_{X_{t_1}}^X\big[\exp - \sum_{i=2}^n \langle X_{t_i - t_1}, \varphi_i\rangle\big]\Big] \\ &= \mathbb{E}_\eta^X\big[\exp(-\langle X_{t_1}, \varphi_1\rangle - \langle X_{t_1}, \tilde{u}(0,\cdot)\rangle)\big],\end{aligned}$$

where by induction hypothesis $\tilde{u}$ solves

$$\tilde{u}(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, \tilde{u}^2(r+s, W_r)\Big] = \sum_{i=2}^n \mathbf{1}_{\{s\le t_i - t_1\}} \mathbb{E}_x\Big[\varphi_i(W_{t_i - t_1 - s})\Big].$$

Using our computations for $n = 1$, we get that

$$\mathbb{E}_\eta^X\Big[\exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle\Big] = \exp - \langle \eta, \bar{u}(0,\cdot)\rangle,$$

where $\bar{u}$ solves

$$\bar{u}(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, \bar{u}^2(r+s, W_r)\Big] = \mathbf{1}_{\{s\le t_1\}} \mathbb{E}_x\big[\varphi_1(W_{t_1-s}) + \tilde{u}(0, W_{t_1-s})\big].$$

Defining the function $u$ by

$$u(t,x) := \mathbf{1}_{\{s\le t_1\}} \bar{u}(s,x) + \mathbf{1}_{\{s>t_1\}} \tilde{u}(s-t_1, x)$$

we have

$$\mathbb{E}_\eta^X\Big[\exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle\Big] = \exp - \langle \eta, u(0,\cdot)\rangle,$$

and it is straightforward to see that $u$ solves (3.5).

To prove uniqueness, observe that if $u$ and $u'$ are both non-negative solutions of (3.5), then $u(s,x) = u'(s,x) = 0$ for all $s \ge T$. Moreover, both $u$ and $u'$ are bounded by $\sum_{i=1}^n \|\varphi_i\|_\infty$ and we obtain

$$\begin{aligned}
\|u - u'\|_\infty &\le \sup_{x\in\mathbb{R}^d, s\ge 0} \mathbb{E}_x\Big[\int_0^T dA_r\, |u^2(r+s, W_r) - u'^2(r+s, W_r)|\Big] \\
&\le 2a_T \sum_{i=1}^n \|\varphi_i\|_\infty\, \|u - u'\|_\infty,
\end{aligned}$$

and hence $u = u'$ as we assume $2a_T \sum_{i=1}^n \|\varphi_i\|_\infty < 1$. □

*Let us now start with the proof of Theorem 1.7 (a).*

We first show that the measure valued process $Z = (Z_t, t \ge 0)$ defined by (3.4) has the same finite-dimensional distributions as a $\sigma$-catalytic super-Brownian motion. We write $\mathbb{P}_\eta^Z$ for the law of $Z$ starting in $Z_0 = \eta$. Let $n \in \mathbb{N}$ and let us fix $0 < t_1 < \ldots < t_n < T$

and $\varphi_1, \dots \varphi_n$ non-negative, continuous and bounded such that $a_T \sum_{i=1}^n \|\varphi_i\|_\infty < 1$. Using (3.48) we can rewrite the Laplace functional of $Z$ as

$$\mathbb{E}_\eta^Z\Big[\exp - \sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle\Big] = \mathbb{E}_{\nu_\eta}^U\Big[\exp - \Big(\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \Gamma_\sigma, \phi\rangle\Big)\Big], \qquad (3.6)$$

where

$$\phi(s,x) := \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}} f(x)\, H^x[\varphi_i(\omega_{t_i-s})].$$

By Lemma 2.16, we have

$$\mathbb{E}_{\nu_\eta}^U\big[\exp - \langle \Gamma_\sigma, \phi\rangle\big] = \exp - \langle \nu_\eta, w\rangle,$$

where $w \in \mathcal{B}_+(\mathbb{R}_+ \times \mathcal{S})$ is a non-negative solution of

$$w(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, w^2(r+s, W_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty dA_r\, \phi(r+s, W_r)\Big]. \qquad (3.9)$$

Using the definition of $\phi$ and $f$, and the excursion formula (3.2),

$$\begin{aligned}
\mathbb{E}_x\Big[\int_0^\infty dA_r\, \phi(r+s, W_r)\Big] &= \sum_{i=1}^n \mathbb{E}_x\Big[\int_0^\infty dA_r\, \mathbf{1}_{\{r+s<t_i\}} H^{W_r}[\varphi_i(\omega_{t_i-r-s})] f(W_r)\Big] \\
&= \sum_{i=1}^n \mathbb{E}_x\Big[\int_0^\infty dL_r\, \mathbf{1}_{\{r<t_i-s\}} H^{W_r}[\varphi_i(\omega_{t_i-r-s})]\Big] \\
&= \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}} \big(\mathbb{E}_x[\varphi_i(W_{t_i-s})] - Q_{t_i-s}\varphi_i(x)\big).
\end{aligned}$$

So, altogether, we have just shown that $w : \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}_+$ solves

$$\begin{aligned}
&w(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, w^2(r+s, W_r)\Big] \\
&\quad = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}} \Big(\mathbb{E}_x\big[\varphi_i(W_{t_i-s})\big] - \mathbb{E}_x\big[\mathbf{1}_{\{t_i<s+\tau_S\}}\, \varphi_i(W_{t_i-s})\big]\Big). \qquad (3.10)
\end{aligned}$$

Let $X$ be a $\sigma$-catalytic super-Brownian motion with start in $X_0 = \eta$. According to Lemma 3.8, we can rewrite the dual equation of $X$ in its backward form, i.e.

$$\mathbb{E}_\eta^X\Big[\exp - \sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle\Big] = \exp - \langle \eta, u(0,\cdot)\rangle,$$

where $u$ is the unique non-negative solution of

$$u(s,x) + \mathbb{E}_x\Big[\int_0^\infty dA_r\, u^2(r+s,W_r)\Big] = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\,\mathbb{E}_x\big[\varphi_i(W_{t_i-s})\big]. \tag{3.11}$$

Define $\tilde{w}(s,x) := w(s,x)$ if $x \in \mathcal{S}$ and

$$\tilde{w}(s,x) := \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\,\mathbb{E}_x\big[\varphi_i(W_{t_i-s})\big] - \mathbb{E}_x\Big[\int_0^\infty dA_r\, w^2(r+s,W_r)\Big],$$

if $x \notin \mathcal{S}$. Then $\tilde{w}$ is a solution of (3.11) and as $\tilde{w}$ is non-negative and (3.11) has a unique non-negative solution, we get $\tilde{w} = u$. In particular, for all $x \in \mathcal{S}$ we have $w(x,s) = u(x,s)$ for all $s \geq 0$.

Using the definition of $\nu_\eta$, the fact that $w$ and $u$ agree for all $x \in \mathcal{S}$, equation (3.11) for $u$, and the strong Markov property,

$$\begin{aligned}
\langle \nu_\eta, w\rangle &= \int \eta(dy)\,\mathbb{E}_y[\mathbf{1}_{\{\tau_S<\infty\}} w(\tau_S, W_{\tau_S})] \\
&= \int \eta(dy)\,\mathbb{E}_y[\mathbf{1}_{\{\tau_S<\infty\}} u(\tau_S, W_{\tau_S})] \\
&= \int \eta(dy)\,\Big(\sum_{i=1}^n \mathbb{E}_y\big[\mathbf{1}_{\{\tau_S<t_i\}}\varphi_i(W_{t_i})\big] - \mathbb{E}_y\Big[\int_0^\infty dA_r\, u^2(r,W_r)\Big]\Big) \\
&= \int \eta(dy)\,\Big(-\sum_{i=1}^n \mathbb{E}_y\big[\mathbf{1}_{\{\tau_S\geq t_i\}}\varphi_i(W_{t_i})\big] + u(0,y)\Big),
\end{aligned}$$

and therefore

$$\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, w\rangle = \langle \eta, u(0,\cdot)\rangle.$$

Hence,

$$\begin{aligned}
\mathbb{E}_\eta^Z\Big[\exp -\sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle\Big] &= \exp -\Big(\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, w\rangle\Big) \\
&= \exp -\langle \eta, u(0,\cdot)\rangle \\
&= \mathbb{E}_\eta^X\Big[\exp -\sum_{i=1}^n \langle X_{t_i}, \varphi_i\rangle\Big],
\end{aligned}$$

and we infer that $Z$ has the same finite-dimensional marginals as a $\sigma$-catalytic super-Brownian motion $X$.

Hence, there is a continuous modification $\tilde{Z}$ of $Z$ which is a $\sigma$-catalytic super-Brownian motion. To show that $\tilde{Z}$ and $Z$ are indistinguishable, observe that, by definition of $Z$ and since $\mathcal{S}$ has vanishing Lebesgue measure, $Z_s(\mathcal{S}) = 0$ for all $s > 0$. Hence it is enough to show that $t \mapsto \langle Z_t, \varphi \rangle$ is right-continuous, for every continuous, non-negative $\varphi$ with compact support disjoint from $\mathcal{S}$.

Fix $t > 0$ and $\varphi$ as above. First observe that $t \mapsto \langle \eta, Q_t\varphi \rangle$ is continuous, and it suffices to look at the second term on the right hand side of (3.48). To prepare an application of dominated convergence note that,

$$\begin{aligned}
&\mathbb{E}^U_{\nu_\eta}\Big[\iint \Gamma_\sigma(dr\,dx)\,\mathbf{1}_{\{r<t\}} f(x)\Big] = \mathbb{E}^U_{\nu_\eta}\Big[\int_0^\infty dr \iint U_r(ds\,dx)\,\mathbf{1}_{\{s<t\}} f(x)\Big] \\
&\quad = \int \nu_\eta(dz)\,\mathbb{E}_z\Big[\int_0^\infty dA_r\, f(W_r)\,\mathbf{1}_{\{r<t\}}\Big] = \int \eta(dz)\,\mathbb{E}_z\Big[\mathbb{E}_{W_{\tau_S}}\Big[\int_0^\infty dL_r\,\mathbf{1}_{\{r+\tau_S<t\}}\Big]\Big] \\
&\quad = \int \eta(dz) \int \mu(dy) \int_0^t dr\, p_r(z,y).
\end{aligned}$$

This is finite, as it is bounded by

$$\int \eta(dz) \int \mu(dy) \int_0^\infty dr\ \mathrm{e}^{t-r}\, p_r(z,y) = \mathrm{e}^t \int \eta(dz)\,\mathbb{E}_z\big[\mathrm{e}^{-\tau_S}\big] < \infty.$$

As $\mathcal{S}$ is closed and the support of $\varphi$ is compact, there is an $\varepsilon > 0$ such that the distance $d(\operatorname{supp}\varphi, \mathcal{S}) > \varepsilon$. Denote by $T_\varepsilon$ the first hitting time of $\mathcal{S}^c_\varepsilon := \{x \in \mathbb{R}^d : d(x,\mathcal{S}) > \varepsilon\}$. Recall from [De96, Lemme 8.3] that $\sup_{x\in\mathcal{S}} H^x[T_\varepsilon < \infty] < \infty$. Then, using dominated convergence in the second, and continuity of $t \mapsto H^x[\varphi(\omega_t)]$ in the last step,

$$\begin{aligned}
&\lim_{s\downarrow t} \iint \Gamma_\sigma(dr\,dx)\mathbf{1}_{\{r<s\}} f(x) H^x[\varphi(\omega_{s-r})] \\
&\quad = \lim_{s\downarrow t} \iint \Gamma_\sigma(dr\,dx)\mathbf{1}_{\{r<s\}} f(x) H^x[\mathbf{1}_{\{T_\varepsilon<\infty\}}\,\varphi(\omega_{s-r})] \\
&\quad = \iint \Gamma_\sigma(dr\,dx) \lim_{s\downarrow t} \mathbf{1}_{\{r<s\}} f(x) H^x[\mathbf{1}_{\{T_\varepsilon<\infty\}}\,\varphi(\omega_{s-r})] \\
&\quad = \iint \Gamma_\sigma(dr\,dx)\mathbf{1}_{\{r<t\}} f(x) H^x[\mathbf{1}_{\{T_\varepsilon<\infty\}}\,\varphi(\omega_{t-r})].
\end{aligned}$$

This shows that, almost surely, $s \mapsto \langle Z_s, \varphi \rangle$ is right-continuous, which completes the proof. $\square$

### 3.2.2 Proof of Theorem 1.7 (b)

We need the following elementary estimate on the heat kernel. For a proof see [De96, Lemma 2.1]

**Lemma 3.9.** *Let $\delta > 0$, then there is a constant $c > 0$ such that for all $x \in \mathbb{R}^d$ and all $s > 0, t > 0$,*

$$|p_s(x) - p_t(x)| \le c \int_{[s,t]} du \, \frac{1}{u} p_{u(1+\delta)}(x).$$

We have to show that the collision local time $\mathcal{L}_{\sigma,Z}$ of $Z$ with the catalytic measure $\sigma$ defined by (1.11) is indeed given by the measure $\Gamma_\sigma$. Let $\psi \in H_b$ and define

$$\langle \mathcal{L}^\varepsilon_{\sigma,Z}, \psi \rangle := \int_0^\infty ds \int \sigma(dy) \int Z_s(dz) \, p_\varepsilon(z-y)\psi(s,y), \tag{3.12}$$

where $Z$ is the $\sigma$-catalytic super-Brownian motion given by

$$\langle Z_s, \varphi \rangle = \langle \eta, Q_t\varphi \rangle + \iint \Gamma_\sigma(dr\, dx) \, \mathbf{1}_{\{r<s\}} f(x) H^x[\varphi(\omega_{s-r})]. \tag{3.13}$$

By Theorem 1.5, there is a random measure $\mathcal{L}_{\sigma,Z}$ called the collision local time of the catalytic super-Brownian motion $Z$ with its catalyst $\sigma$, such that almost surely,

$$\lim_{\varepsilon \downarrow 0} \langle \mathcal{L}^\varepsilon_{\sigma,Z}, \psi \rangle = \langle \mathcal{L}_{\sigma,Z}, \psi \rangle.$$

Moreover, the convergence also holds in $L^p(\mathbb{P}^Z_\eta)$ for any $p \ge 1$. To deduce that $\mathcal{L}_{\sigma,Z}$ is given by the random measure $\Gamma_\sigma$, it is enough to show

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}^Z_\eta \left[ \langle \mathcal{L}^\varepsilon_{\sigma,Z} - \Gamma_\sigma, \psi \rangle^2 \right] = 0, \tag{3.14}$$

as convergence in $L^2(\mathbb{P}^Z_\eta)$ implies the $\mathbb{P}^Z_\eta$ almost sure convergence of a subsequence. Using (3.12), (3.13) and Fubinis Theorem,

$$\langle \mathcal{L}^\varepsilon_{\sigma,Z} - \Gamma_\sigma, \psi \rangle = \int_0^\infty ds \int \sigma(dy)\psi(s,y)\langle \eta, Q_s p_\varepsilon(y,\cdot)\rangle \; + \; \langle \Gamma_\sigma, \phi_\varepsilon \rangle,$$

where the function $\phi_\varepsilon \in \mathcal{B}(\mathbb{R}_+ \times \mathcal{S})$ is given by

$$\phi_\varepsilon(r,x) := \int_0^\infty ds \int \sigma(dy) \, \psi(y,s) \mathbf{1}_{\{r<s\}} f(x) H^x \left[ p_\varepsilon(\omega_{s-r}, y) \right] \; - \; \psi(r,x).$$

Hence, we obtain

$$\mathbb{E}^Z_\eta\big[\langle \mathcal{L}^\varepsilon_{\sigma,Z} - \Gamma_\sigma, \psi\rangle^2\big] = \Big[\int_0^\infty ds \int \sigma(dy)\, \psi(s,y) \langle \eta, Q_s p_\varepsilon(y,\cdot)\rangle\Big]^2$$
$$+ 2\int_0^\infty ds \int \sigma(dy)\, \psi(s,y)\langle \eta, Q_s p_\varepsilon(y,\cdot)\rangle\; \mathbb{E}^Z_\eta\big[\langle \Gamma_\sigma, \phi_\varepsilon\rangle\big]$$
$$+ \mathbb{E}^Z_\eta\big[\langle \Gamma_\sigma, \phi_\varepsilon\rangle^2\big]. \tag{3.15}$$

We show that all three terms in (3.15) tend to 0 as $\varepsilon \downarrow 0$.

1° *First summand.*
Denote by $p^*_s(x,y)$ the transition density of Brownian motion killed on $\mathcal{S}$. We have to show that

$$\lim_{\varepsilon\downarrow 0} \int_0^\infty ds \int \sigma(dy)\, \psi(s,y) \int \eta(dx) \int dz\, p_\varepsilon(y,z) p^*_s(x,z) = 0. \tag{3.16}$$

To prepare an application of dominated convergence, we show that

$$\int \sigma(dy)\, \psi(s,y) \int dz\, p_\varepsilon(y,z) p^*_s(x,z)$$

is bounded by an $\int_0^\infty ds \int \eta(dx)$-integrable function. As $\psi \in H_b$ it is bounded with $\operatorname{supp}\psi \in [0,T]\times\mathbb{R}^d$ for some $T>0$. Moreover, as $p^*_s(x,z) \le p_s(x,z)$, we can estimate using the Chapman Kolmogorov equation and Lemma 2.4,

$$\int \sigma(dy)\, \psi(s,y) \int dz\, p_\varepsilon(y,z) p^*_s(x,z) \le \int \sigma(dy)\, \psi(s,y) \int dz\, p_\varepsilon(y,z) p_s(x,z)$$
$$= \int \sigma(dy)\, \psi(s,y) p_{s+\varepsilon}(x,y)$$
$$\le c\|\psi\|_\infty \frac{1}{(s\wedge 1)^{1-\beta}} \mathbf{1}_{[0,T]}(s), \tag{3.17}$$

for some $c>0$, and where $\beta \in (0,1)$ is the constant refering to $\sigma$ in (1.2). As $\eta$ is a finite measure, the right-hand-side of (3.17) is a $\int_0^\infty ds \int \eta(dx)$-integrable function and by dominated convergence it remains to show that

$$\lim_{\varepsilon\downarrow 0} \int \sigma(dy) \int dz\, p_\varepsilon(y,z) p^*_s(x,z) = 0, \tag{3.18}$$

in order to deduce (3.16). Notice that we can apply dominated convergence to the outer integral by similar arguments as before, in particular, using $p^*_s(x,z) \le p_s(x,z)$, the Chapman Kolmogorov equation and Lemma 2.4. Moreover, it is a classical result on killed Brownian motion that the function $z \mapsto p_s^*(x,z)$ is continuous and vanishes on $\mathcal{S}$ (see e.g. [CZ95, Chapter 2.2]). Hence, we obtain (3.18).

$2^\circ$ *Second summand.*
As by Theorem 2.14, the random measure $\Gamma_\sigma$ is distributed as the collision local time we can use Lemma 2.20 to compute the moments of $\Gamma_\sigma$. Hence, we get by an easy calculation, also using that $dL_r = dA_r\, f(W_r)$,

$$\begin{aligned}\mathbb{E}^Z_\eta\left[\langle \Gamma_\sigma, \phi_\varepsilon\rangle\right] = \int \eta(dx) \int_0^\infty ds \int \sigma(dy)\, \psi(s,y) \mathbb{E}_x\Big[\int_0^\infty dL_r\, \mathbf{1}_{\{r<s\}} H^{W_r}\left[p_\varepsilon(\omega_{s-r}, y)\right]\Big] \\ - \mathbb{E}^Z_\eta\left[\langle \Gamma_\sigma, \psi\rangle\right].\end{aligned}$$

By the last exit decomposition of the Brownian motion (3.2), we have

$$\mathbb{E}_x\Big[\int_0^\infty dL_r\, \mathbf{1}_{\{r<s\}} H^{W_r}\left[p_\varepsilon(\omega_{s-r}, y)\right]\Big] = \mathbb{E}_x\left[p_\varepsilon(W_s, y)\right] - Q_s p_\varepsilon(\cdot, y)(x).$$

Hence, using part $1^\circ$ of the proof is suffices to show that

$$\int \eta(dx) \int_0^\infty ds \int \sigma(dy)\, \psi(s,y) \mathbb{E}_x\left[p_\varepsilon(W_s, y)\right]$$

converges to $\mathbb{E}^Z_\eta[\langle \Gamma_\sigma, \psi\rangle]$. Using the Chapman-Kolmogorov equation for the Brownian transition density, we can rewrite the last displayed equation to

$$\int \eta(dx) \int_0^\infty ds \int \sigma(dy)\, \psi(s,y) p_{s+\varepsilon}(x,y).$$

As $\psi$ is in $H_b$ it is bounded with $\operatorname{supp}\psi \in [0,T] \times \mathbb{R}^d$ for some $T > 0$. Moreover, as $\sigma$ is uniformly non-polar, we have by Lemma 2.4 that there is a constant $c > 0$ such that

$$\int \sigma(dy) \psi(s,y) p_{s+\varepsilon}(x,y) \le c \|\psi\|_\infty \frac{1}{(s \wedge 1)^{1-\beta}} \mathbf{1}_{[0,T]}(s), \tag{3.19}$$

where $\beta \in (0,1)$ is the constant refering to $\sigma$ in (1.2). Hence, as $\eta$ is a finite measure, the left hand side of equation (3.19) is $\int \eta(dx) \int_0^\infty ds$-integrable and therefore we can apply dominated convergence and hence it only remains to show that

$$\lim_{\varepsilon \downarrow 0} \int \sigma(dy) \psi(s,y) p_{s+\varepsilon}(x,y) = \int \sigma(dy) \psi(s,y) p_s(x,y).$$

Let $\delta > 0$. By Lemma 3.9 and Lemma 2.4 there are constants $c > 0$ and $c' > 0$ such that

$$\int \sigma(dy)\, \psi(s,y)|p_{s+\varepsilon}(x,y) - p_s(x,y)| \le c \int \sigma(dy) \int_s^{s+\varepsilon} du\, \frac{1}{u} p_{u(1+\delta)}(x,y)$$
$$\le c' \int_s^{s+\varepsilon} du\, \frac{1}{u} \cdot \frac{1}{(u(1+\delta)\wedge 1)^{1-\beta}},$$

which tends to 0 as $\varepsilon \downarrow 0$.

3° *Third summand.*
Using the formula for the second moment of the collision local time, we obtain

$$\mathbb{E}_\eta^Z\left[\langle \Gamma_\sigma, \phi_\varepsilon \rangle^2\right] = \mathbb{E}_\eta^Z\left[\langle \Gamma_\sigma, \phi_\varepsilon \rangle\right]^2 + \mathbb{E}_\eta^Z\Big[\int_0^\infty dA_r\, \mathbb{E}_{\delta_{B_r}}^Z\left[\langle \Gamma_\sigma, \phi_\varepsilon(r+\cdot,\cdot)\rangle\right]^2\Big].$$

Using part 2°, we only have to deal with the second summand which is equal to

$$\int \eta(dx) \int_0^\infty dr \int \sigma(dy)\, p_r(x,y) \Big[\mathbb{E}_{\delta_y}^Z\left[\langle \Gamma_\sigma, \phi_\varepsilon(r+\cdot,\cdot)\rangle\right]\Big]^2.$$

By exactly the same arguments as in part 2°, there is a constant $c > 0$ such that

$$\mathbb{E}_{\delta_y}^Z\left[\langle \Gamma_\sigma, \phi_\varepsilon(r+\cdot,\cdot)\rangle\right] \le c \int_0^\infty ds\, \frac{1}{[(s-r)\wedge 1]^{1-\beta}} \mathbf{1}_{[0,T]}(s-r).$$

Moreover, there is a $c' > 0$ such that

$$\int \sigma(dy)\, p_r(x,y) \le c \frac{1}{(r\wedge 1)^{1-\beta}}.$$

Hence, by dominated convergence and part 2° also the third summand tends to 0. □

### 3.2.3 Smoothness of the density field

The following corollary to Theorem 1.7 is already contained in [De96]. DELMAS proved the smoothness of the density field from his representation of $\sigma$-catalytic super-Brownian motion given by equation (1.19). We can use essentially the same arguments to deduce this result from Theorem 1.7. Of course, we have to make the same assumptions as in Theorem 1.7. Hence, DELMAS result is more general than ours, as he only needs to assume that the capacitary measure $\mu$ on $\mathcal{S}$ is uniformly non-polar.

Assume that $\sigma$ is a uniformly non-polar measure such that its fine support $\mathcal{S}$ has vanishing Lebesgue measure. Moreover, let the capacitary measure $\mu$ of $\mathcal{S}$ be absolutely continuous with respect to $\sigma$, i.e. there is a density $f$ such that $\mu(dx) = f(x)\sigma(dx)$. Let $Z = (Z_t, t \geq 0)$ be a $\sigma$-catalytic super-Brownian motion. Under these assumptions, we show the following:

**Corollary 3.10** (***Smoothness of the density field).***
*Almost surely, for all $t > 0$ the random measure $Z_t$ on $\mathcal{S}^c := \mathbb{R}^d \setminus \mathcal{S}$ has a density $z_t(y)$ with respect to the Lebesgue measure. Moreover, $z$ is in $\mathcal{C}^\infty\big((0,\infty) \times \mathcal{S}^c\big)$ and solves the heat equation outside the catalyst, meaning that $\partial_t\, z_t(y) = \frac{1}{2}\Delta_y\, z_t(y)$ for all $t \in (0,\infty)$, $y \in \mathcal{S}^c$.*

Hence, the density field $z(t,y)$ solves the heat equation on $(0,\infty) \times \mathcal{S}^c$ with generalized initial condition on $\{t = 0\}$ given by the measure $\eta$. Indeed, the continuity of $Z$ in the weak topology implies, that we have for all $\varphi \in \mathcal{C}^b(\mathbb{R}_+)$, almost surely

$$\lim_{t \downarrow 0} \int dy\, \varphi(y) z_t(y) \;=\; \int \eta(dy)\, \varphi(y).$$

Moreover, the generalized boundary condition on $(0,\infty) \times \mathcal{S}$ is given by the collision local time $\Gamma_\sigma$, i.e. we have by Theorem 1.7(b) that for all $\psi \in H_b$,

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty dt \int \sigma(dy) \int dx\, z_t(x) p_\varepsilon(x,y)\psi(t,y) = \iint \Gamma_\sigma(dr\, dx)\psi(r,x).$$

*Proof of Corollary 3.10.* (following closely [De96]) Recall that we denote by $\tau_{\mathcal{S}}$ the first hitting time of $\mathcal{S}$ and denote by $p^*_t(x,y)$ the transition density of Brownian motion killed on $\mathcal{S}$, i.e. for all $t \in (0,\infty), x, y \in \mathbb{R}^d$,

$$p^*_t(x,y) = p_t(x,y) - \mathbb{E}_x\big[\mathbf{1}_{\{\tau_{\mathcal{S}} < t\}} p_t(t - \tau_{\mathcal{S}}, W_{\tau_{\mathcal{S}}} - y)\big]. \tag{3.20}$$

It is well known, that for all $x \in \mathbb{R}^d$, the function $(t,y) \mapsto p^*_t(x,y)$ is in $\mathcal{C}^\infty\big((0,\infty) \times \mathcal{S}^c\big)$ and solves the heat equation, i.e.

$$\partial_t p^*_t(x,y) = \tfrac{1}{2}\Delta_y p^*_t(x,y), \qquad \text{for } t \in (0,\infty), y \in \mathcal{S}^c. \tag{3.21}$$

Moreover, we deduce by induction from (3.20) that for all $n \in \mathbb{N}, m \in \mathbb{N}^d$, $\varepsilon > 0$ and all $\delta_0 > 0$, we have

$$\sup\big\{\big|\partial_t^n \partial_y^m p^*_t(x,y)\big| : t > 0, x \in \mathbb{R}^d, y \in \mathcal{S}_\varepsilon, |x - y| > \delta_0\big\} < \infty, \tag{3.22}$$

where $\mathcal{S}_\varepsilon := \{x \in \mathbb{R}^d : d(x,\mathcal{S}) \leq \varepsilon\}$. Define $\tau_\varepsilon$ to be the first hitting time of $\mathcal{S}^c_\varepsilon$. Let $\varphi \in \mathcal{B}_+(\mathbb{R}^d)$ be supported in $\mathcal{S}^c_\varepsilon$. Hence, using the strong Markov property of the excursion measure with respect to the semigroup of Brownian motion killed on $\mathcal{S}$, we obtain for all $x \in \mathcal{S}$ and all $s < t$,

$$
\begin{aligned}
H^x\big[\varphi(\omega_{t-s})\big] &= H^x\Big[\int dy\,\varphi(y)\mathbf{1}_{\{t-s-\tau_\varepsilon>0\}}p^*_{t-s-\tau_\varepsilon}(\omega_{\tau_\varepsilon},y)\Big] \\
&= \int dy\,\varphi(y)H^x\big[\mathbf{1}_{\{t-s-\tau_\varepsilon>0\}}p^*_{t-s-\tau_\varepsilon}(\omega_{\tau_\varepsilon},y)\big].
\end{aligned} \tag{3.23}
$$

In particular, we deduce from the representation of the $\sigma$-catalytic super-Brownian motion in Theorem 1.7 that it possesses a density on $\mathcal{S}^c_\varepsilon$ which is given explicitly by

$$
z^\varepsilon_t(y) := \int \eta(dx)\,p^*_t(x,y) + \iint \Gamma_\sigma(ds,dx)\,f(x)H^x\big[\mathbf{1}_{\{t-s-\tau_\varepsilon>0\}}p^*_{t-s-\tau_\varepsilon}(\omega_{\tau_\varepsilon},y)\big]. \tag{3.24}
$$

Notice that by (3.22) and since $(t,y) \mapsto p^*_t(x,y)$ is in $\mathcal{C}^\infty\big((0,\infty)\times\mathcal{S}^c\big)$ the function $(t,y) \mapsto \int \eta(dx)\,p^*_t(x,y)$ is also $\mathcal{C}^\infty$. Justifying dominated convergence as in the proof of Theorem 1.7(b) shows that the second summand of (3.24) as a function of $t$ and $y$ is continuous (outside a set of zero probability). Using induction and (3.22) one checks that it is even $\mathcal{C}^\infty$ on $(0,\infty)\times\mathcal{S}^c$. Finally, observe that (3.21) implies that $(t,y) \mapsto z^\varepsilon_t(y)$ solves the heat equation on $(0,\infty)\times\mathcal{S}^c$ which completes the proof. □

### 3.2.4 The absolute continuity condition

In Theorem 1.7 we have to assume the abstract condition, that the capacitary measure $\mu$ on the set $\mathcal{S}$ is *absolutely continuous* with respect to $\sigma$, i.e. that there is a density $f$ such that

$$
\mu(dx) = \sigma(dx)\,f(x).
$$

Recall that by the Radon-Nikodym theorem, $\mu$ is absolutely continuous with respect to $\sigma$, if and only if for every $N \in \mathcal{B}(\mathbb{R}^d)$ with $\sigma(N) = 0$ we also have $\mu(N) = 0$. If there is a Borel set $M$ such that $\sigma(M) = 0$ but $\mu(M) > 0$, then we call $\mu$ *singular* with respect to $\sigma$.

**Proposition 3.11 (Finite point measures).**
*Let $\sigma \in \mathcal{M}^p_f(\mathbb{R})$ be a finite point measure in one dimension and $\mathcal{S} := \operatorname{supp}\sigma$. Then the capacitary measure $\mu$ on $\mathcal{S}$ is absolutely continuous with respect to $\sigma$.*

*Proof.* Let $n \in \mathbb{N}$ and $\alpha_1,\ldots,\alpha_n > 0$. Moreover, let $c_1,\ldots,c_n \in \mathbb{R}$ and $\sigma$ be of the form

$$
\sigma = \sum_{i=1}^n \alpha_i\delta_{c_i},
$$

where $\delta_c$ denotes the Dirac point mass at $c \in \mathbb{R}$. Hence, $\mathcal{S} = \{c_1,\ldots,c_n\}$. Assume that $\sigma(N) = 0$ for some $N \subseteq \mathbb{R}$. Then $N \cap \mathcal{S} = \emptyset$. Let $\mu$ be the capacitary measure on $\mathcal{S}$.

Then by Proposition 3.3,

$$\mu(N) = \int_{\mathbb{R}^d} dx\, \mathbb{E}_x\big[\mathrm{e}^{-\tau_S}\, \mathbf{1}_N(W_{\tau_S})\big] \leq \int_{\mathbb{R}^d} dx\, \mathbb{P}_x(W_{\tau_S} \in N),$$

where $\tau_S$ is the first hitting time of $\mathcal{S}$. As $W_{\tau_S} \in \mathcal{S}$ almost surely and $N \cap \mathcal{S} = \emptyset$, we have $\mathbb{P}_x(W_{\tau_S} \in N) = 0$ and hence $\mu(N) = 0$. And therefore $\mu$ is absolutely continuous with respect to $\sigma$ by the Radon-Nikodym Theorem. $\square$

Indeed, we can generalize the proof of Proposition 3.11 in the following way: for the set $\mathcal{S} \subseteq \mathbb{R}^d$ and $x \in \mathcal{S}^{\mathrm{c}}$, we define the *harmonic measure* of $\mathcal{S}$ starting in $x$ by

$$\mathfrak{h}^x(dy) := \mathbb{P}_x(W_{\tau_S} \in dy).$$

For a measure $\sigma \in \mathcal{M}(\mathbb{R}^d)$ with fine support $\mathcal{S}$, the following proposition gives a sufficient condition for the capacitary measure $\mu$ of $\mathcal{S}$ being absolutely continuous with respect to $\sigma$.

**Proposition 3.12.** *Let $\sigma \in \mathcal{M}(\mathbb{R}^d)$ and $\mu$ be the capacitary measure of $\mathcal{S} := \operatorname{supp}\sigma$. Moreover, denote by $\mathfrak{h}^x$ the harmonic measure of $\mathcal{S}$ starting in $x \in \mathcal{S}^{\mathrm{c}}$. Suppose that, $\mathcal{S}$ has vanishing $d$-dimensional Lebesgue measure, and that the harmonic measure $\mathfrak{h}^x$ is absolutely continuous with respect to $\sigma$ for all $x \in \mathcal{S}^{\mathrm{c}}$. Then, $\mu$ is absolutely continuous with respect to $\sigma$.*

*Proof.* Let $N \in \mathcal{B}(\mathbb{R}^d)$ such that $\sigma(N) = 0$. If $\mathfrak{h}^x$ is absolutely continuous with respect to $\sigma$ for any $x \in \mathcal{S}^{\mathrm{c}}$ we have $\mathfrak{h}^x(N) = 0$ for any $x \in \mathcal{S}^{\mathrm{c}}$. Hence, by Proposition 3.3 and as $\mathcal{S}$ has vanishing $d$-dimensional Lebesgue measure,

$$\begin{aligned}
\mu(N) &= \int_{\mathbb{R}^d} dx\, \mathbb{E}_x\big[\mathrm{e}^{-\tau_S}\, \mathbf{1}_N(W_{\tau_S})\big] \\
&\leq \int_{\mathbb{R}^d} dx\, \mathbb{E}_x\big[\mathbf{1}_N(W_{\tau_S})\big] \\
&= \int_{\mathbb{R}^d \setminus \mathcal{S}} dx\, \mathfrak{h}^x(N) \;=\; 0,
\end{aligned}$$

and therefore $\mu$ is absolutely continuous with respect to $\sigma$. $\square$

**Corollary 3.13 (The surface measure on a smooth manifold).**
*Let $\sigma$ be the surface measure on a smooth $d-1$ dimensional manifold $\mathcal{S}$ in $\mathbb{R}^d, d \geq 2$, then the capacitary measure $\mu$ on $\mathcal{S}$ is absolutely continuous with respect to $\sigma$.*

*Proof.* Clearly, $\mathcal{S}$ has vanishing $d$-dimensional Lebesgue measure. Moreover, it is well known that for any $x \in \mathcal{S}^{\mathrm{c}}$ the harmonic measure $\mathfrak{h}^x$ on $\mathcal{S}$ is absolutely continuous with respect to the surface measure $\sigma$ on $\mathcal{S}$ (see e.g. [Ba95, Chapter III.5]). Hence, the assertion follows from Proposition 3.12. $\square$

Moreover, for the smooth manifold case, it is possible to construct the density of $\mu$ with respect to $\sigma$ explicitly: From similar arguments to those in [Ba95, Proposition II.3.11], there is a constant $c_d$ (depending only on $d$), such that

$$\mathbb{E}_x\left[\mathrm{e}^{-\tau_S}\,\psi(W_{\tau_S})\right] = c_d \int_{\mathcal{S}} \sigma(dy)\,\psi(y)\partial_{n_y} g^{1,*}(x,y),$$

where $\partial_n$ denotes the outward normal derivative on $\mathcal{S}$, $g^{1,*}(x,y) = \int_0^\infty dt\, \mathrm{e}^{-t}\, p_t^*(x,y)$ and $p_t^*(x,y)$ is the density of Brownian motion killed on $\mathcal{S}$. Hence, for any Borel set $A \in \mathcal{B}(\mathcal{S})$,

$$\mu(A) = c_d \int_A \sigma(dy) \int_{\mathbb{R}^d} dx\, \partial_{n_y} g^{1,*}(x,y).$$

To see a particular example, let $\mathcal{S}$ be the boundary of an open ball in $\mathbb{R}^d, d \geq 2$ with radius $r > 0$. Then we deduce from [PS78, Prop 1.9] that

$$\mu(A) = \frac{2\pi^{d/2} r^{d-2}}{\Gamma(\frac{d}{2}-1)}\,\sigma(A),$$

for any Borel set $A \in \mathcal{B}(\mathcal{S})$. Notice that the density is a constant but depends on the curvature of $\mathcal{S}$.

The following proposition gives a large set of counter examples where the absolute continuity condition fails: always when $\sigma$ is absolutely continuous with respect to the $d$-dimensional Lebesgue measure. However, these measures fail the condition of having vanishing $d$-dimensional Lebesgue measure anyway.

**Proposition 3.14.** *Let $\sigma \in \mathcal{M}(\mathbb{R}^d)$ be a non-zero measure which is absolutely continuous with respect to d-dimensional Lebesgue measure $\lambda^d$. Moreover assume that the boundary $\partial\mathcal{S}$ is non-polar for the Brownian motion. Then the capacitary measure $\mu$ on $\mathcal{S}$ is singular with respect to $\sigma$.*

*Proof.* Assume that $\sigma \in \mathcal{M}(\mathbb{R}^d)$ be absolutely continuous with respect to $\lambda^d$. If we denote by $\partial\mathcal{S}$ the boundary of $\mathcal{S}$, we have $\lambda^d(\partial\mathcal{S}) = 0$. As $\sigma$ is absolutely continuous with respect to $\lambda^d$, we also have $\sigma(\partial\mathcal{S}) = 0$. Denote by $\tau_{\partial\mathcal{S}}$ the first hitting time of $\partial\mathcal{S}$. Then it follows (see e.g. [PS78, Theorem 2.6.5] for the second equality) that, for some fixed $t > 0$, ($\overline{\mathcal{S}}$ denotes the Euclidian closure of $\mathcal{S}$),

$$\begin{aligned}
\mu(\partial\mathcal{S}) &= \int dx\, \mathbb{E}_x\big[\mathbf{1}_{\{\tau_\mathcal{S}<\infty\}}\, \mathrm{e}^{-\tau_\mathcal{S}}\, \mathbf{1}_{\partial\mathcal{S}}(W_{\tau_\mathcal{S}})\big] \\
&\geq \int_{(\overline{\mathcal{S}})^c} \mathbb{E}_x\big[\mathbf{1}_{\{\tau_{\partial\mathcal{S}}<\infty\}}\, \mathrm{e}^{-\tau_{\partial\mathcal{S}}}\, \mathbf{1}_{\partial\mathcal{S}}(W_{\tau_{\partial\mathcal{S}}})\big] \\
&\geq \int_{(\overline{\mathcal{S}})^c} dx\, \mathrm{e}^{-t}\, \mathbb{P}_x\big(\tau_{\partial\mathcal{S}} \leq t, W_{\tau_{\partial\mathcal{S}}} \in \partial\mathcal{S}\big) \\
&= \int_{(\overline{\mathcal{S}})^c} dx\, \mathrm{e}^{-t}\, \mathbb{P}_x(\tau_{\partial\mathcal{S}} \leq t),
\end{aligned}$$

as $W_{\tau_{\partial\mathcal{S}}} \in \partial\mathcal{S}$ almost surely. Hence, $\mu(\partial\mathcal{S}) > 0$ because we assumed $\partial\mathcal{S}$ to be non-polar. □

Of course, if $\sigma$ is a non-zero measure on $\mathbb{R}^d$ which is absolutely continuous with respect to $\lambda^d$, it follows that $\lambda^d(\mathcal{S}) > 0$. One might ask, if for any measure $\sigma \in \mathcal{M}(\mathbb{R}^d)$ such that $\lambda^d(\mathcal{S}) > 0$, the capacitary measure on $\mathcal{S}$ is singular with respect to $\sigma$. That is indeed *not* the case as we see by the following example: Let $\sigma \in \mathcal{M}(\mathbb{R})$ be given by $\sigma = \lambda^1_{(0,1)} + \delta_0 + \delta_1$, where $\lambda^1_{(0,1)}$ is the restriction of the one-dimensional Lebesgue measure to the interval $(0,1)$. Then $\mathcal{S} = [0,1]$ and $\lambda^1(\mathcal{S}) > 0$. But on the other hand, it is trivial to check by Proposition 3.3 that the capacitary measure $\mu$ on $\mathcal{S}$ is absolutely continuous with respect to $\sigma$.

**Proposition 3.15 (The Hausdorff measure on the Cantor set).**
*Let $\sigma$ be the Hausdorff measure on the classical middle-third Cantor set $C$ and $\mu$ be the capacitary measure on $C$. Then $\mu$ is singular with respect to $\sigma$.*

*Proof.* Let $N := \{2/3\}$. Then $\sigma(N) = 0$. On the other hand, by to Proposition 3.3, for some $t > 0$,

$$\begin{aligned}
\mu(N) &= \int_\mathbb{R} dx\, \mathbb{E}_x\big[\mathrm{e}^{-\tau_\mathcal{S}}\, \mathbf{1}_N(W_{\tau_\mathcal{S}})\big] \\
&\geq \int_{\left(\frac{1}{3},\frac{2}{3}\right)} dx\, \mathbb{E}_x\big[\mathrm{e}^{-\tau_\mathcal{S}}\, \mathbf{1}_{\{\frac{2}{3}\}}(W_{\tau_\mathcal{S}})\big] \\
&= \int_{\left(\frac{1}{3},\frac{2}{3}\right)} dx\, \Big(\mathbb{E}_x\big[\mathbf{1}_{\{t<\tau_\mathcal{S}\}}\, \mathrm{e}^{-\tau_\mathcal{S}}\, \mathbf{1}_{\{2/3\}}(W_{\tau_\mathcal{S}})\big] + \mathbb{E}_x\big[\mathbf{1}_{\{t>\tau_\mathcal{S}\}}\, \mathrm{e}^{-\tau_\mathcal{S}}\, \mathbf{1}_{\{2/3\}}(W_{\tau_\mathcal{S}})\big]\Big) \\
&\geq \int_{\left(\frac{1}{3},\frac{2}{3}\right)} dx\, \mathrm{e}^{-t}\, \mathbb{P}_x(t > \tau_\mathcal{S}, W_{\tau_\mathcal{S}} = 2/3),
\end{aligned}$$

which is strictly positive, using e.g. [PS78, Proposition 2.8.3]. □

## 3.3 Reflecting Brownian motion

In this section we discuss the standard reflecting Brownian motion in a bounded domain $D \subseteq \mathbb{R}^d$ with $\mathcal{C}^3$-boundary $\partial D$. The reflecting Brownian motion can be constructed in terms of its transition density $\bar{p}_t(x,y)$ which is the fundamental solution of the heat equation with the homogeneous Neumann boundary condition. Roughly speaking, reflecting Brownian motion behaves like standard Brownian motion inside the domain $D$ and reflects towards the interior along the normal direction each time it reaches the boundary $\partial D$ of $D$.

The section is organized as follows: first in Section 3.3.1 we review the construction of reflecting Brownian motion from its transition density. The following section then reminds the reader of the local time $\ell$ on the boundary $\partial D$. Then in Section 3.3.3 we collect a series of estimates on reflecting Brownian motion we shall need later in the thesis. Finally, Section 3.3.4 deals with excursion theory for reflecting Brownian motion. In particular, we treat the last exit decomposition.

### 3.3.1 Existence and basic properties of the transition density

Denote by $D \subseteq \mathbb{R}^d$ a bounded domain with $\mathcal{C}^3$-boundary $\partial D$, i.e. for every $x \in D$ there exists a local coordinate system such that the boundary looks like a $\mathcal{C}^3$-function. Moreover, denote by $\bar{D}$ the closure of $D$. As we assume the boundary to be $\mathcal{C}^3$, the outward unit normal vector field $n_x$ exists, and the outward unit normal derivative $\partial_n$ of smooth functions $\varphi \in \mathcal{C}^1(\bar{D})$ on $\partial D$ given by

$$\partial_n \varphi(x) := \langle \nabla \varphi(x), n_x \rangle,$$

is well defined. The following theorem can be found in [SU65].

**Theorem 3.16 (The heat equation with homogenous Neumann condition).** *Let $D \subseteq \mathbb{R}^d$ a domain with $\mathcal{C}^3$-boundary. There exists a unique function $\bar{p}_t(x,y)$ defined on $(0,\infty) \times \bar{D} \times \bar{D}$ satisfying the following conditions:*

*(i) $\bar{p}_t(x,y)$ is continuously differentiable in $t > 0$ for fixed $(x,y) \in \bar{D} \times \bar{D}$, and for $\varepsilon > 0$ its derivative is uniformly bounded for $t \geq \varepsilon, x, y \in \bar{D}$. As a function of $x$, $p_t(x,y)$ belongs to $\mathcal{C}^1(\bar{D}) \cap \mathcal{C}^2(D)$ for fixed $t \in (0,\infty), y \in \bar{D}$.*

*(ii) $\bar{p}_t(x,y)$ solves the heat equation inside $D$*

$$\partial_t \bar{p}_t(x,y) = \tfrac{1}{2}\Delta_x \bar{p}_t(x,y) \quad \textit{for } (t,x,y) \in (0,\infty) \times D \times \bar{D}$$

*with the boundary condition*

$$\partial_{n_x}\bar{p}_t(x,y) = 0 \quad \textit{for } (t,x,y) \in (0,\infty) \times \partial D \times \bar{D}.$$

(iii) *Moreover* $\bar{p}_t(x,y)$ *satisfies the initial condition* $\lim_{t\downarrow 0}\bar{p}_t(x,y) = \delta_y(x)$ *this means for any* $\varphi \in \mathcal{C}(\bar{D})$ *and any* $y \in \bar{D}$*, we have*

$$\lim_{t\downarrow 0}\int_{\bar{D}} dx\,\varphi(x)\bar{p}_t(x,y) = \varphi(y).$$

An elegant construction of the function $\bar{p}_t(x,y)$ using the method of *parametrix* can be found in [Hsu84]. If $D$ has some kind of symmetry, then $\bar{p}_t(x,y)$ can be explicitly calculated (e.g. for the case of a unit ball $D = B_1(0), d \geq 2$).

Moreover (still following [SU65]), the function $\bar{p}_t(x,y)$ is strictly positive, symmetric in $x$ and $y$ and satisfies

$$\int_D dy\,\bar{p}_t(x,y) \;=\; 1,$$

and also the *Chapman-Kolmogorov equation*

$$\bar{p}_{t+s}(x,y) = \int_D dz\,\bar{p}_t(x,z)\,\bar{p}_s(z,y).$$

Hence, $\bar{p}_t(x,y)$ is the transition density function of a Markov process $B = (B_t, t \geq 0)$ on $\bar{D}$. As we also have for all $\varepsilon > 0$,

$$\lim_{t\downarrow 0}\frac{1}{t}\sup_{x\in\bar{D}}\int_{O_\varepsilon(x)^{\mathrm{c}}\cap D} dy\,\bar{p}_t(x,y) \;=\; 0, \tag{3.25}$$

where $O_\varepsilon(x)^{\mathrm{c}}$ denotes the complement of a ball around $x$ with radius $\varepsilon > 0$, we can choose a continuous version of the process $B$ (see e.g. [Ch82, Chapter 3.1]). With a slight abuse of notation, let us denote by $\mathbb{P}_x$ its law starting in $x \in \bar{D}$. The continuity of the transition density also ensures that $B$ is a strong Feller process, i.e. that the function $x \mapsto \mathbb{E}_x[f(B_t)]$ is continuous on $\bar{D}$ for all $f \in \mathcal{B}(\bar{D})$. As any Feller process with continuous sample paths is a strong Markov process, we can pass to the following definition:

**Definition 3.17 (Reflecting Brownian motion).**
*The continuous strong Markov process* $B = (B_t, t \geq 0)$ *having the transition density* $\bar{p}_t(x,y)$ *is called the* reflecting Brownian motion *in* $D$*.*

With a slight abuse of notation, we denote by $(\mathcal{F}_t, t \geq 0)$ the right continuous filtration generated by $B$ completed in the usual way.

### 3.3.2 The local time on the boundary

There exists a unique continuous additive functional $\ell = (\ell_t, t \geq 0)$ of $B$ called the *local time* on the boundary $\partial D$, such that for every $\varphi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$ and $x \in \bar{D}$,

$$\mathbb{E}_x\left[\int_0^\infty d\ell_s\, \varphi(s, B_s)\right] = \int_0^\infty ds \int_{\partial D} \sigma(dy)\, \varphi(s,y)\bar{p}_s(x,y), \tag{3.26}$$

where $\sigma$ is the surface measure on $\partial D$. In other words, $\sigma$ is the *Revuz-measure* of the continuous additive functional $\ell$. For $x \in \bar{D}$, let $d(x, \partial D) = \inf\{|x-y| : y \in \partial D\}$. The continuous additive functional $\ell$ can be constructed explicitly as

$$\ell_t = \lim_{n\to\infty} \frac{1}{\varepsilon_n} \int_0^t ds\, \mathbf{1}_{\{x\in\bar{D}\,:\,d(x,\partial D)\leq\varepsilon_n\}}(B_s), \tag{3.27}$$

where the limit exists for all $t \geq 0$, $\mathbb{P}_x$-almost surely for a positive sequence $(\varepsilon_n, n \in \mathbb{N})$ decreasing to 0 which does not depend on $x \in \bar{D}$ (see Theorem 7.2 in [ST62]).

Using the local time on the boundary, we have the following martingale problem characterization of the reflecting Brownian motion (see e.g. [CMK78] for a proof of this result):

**Lemma 3.18 (Martingale problem for reflecting Brownian motion).**
*For every $\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ such that $\Delta\phi$ is bounded on $D$, the process*

$$\phi(B_t) - \phi(B_0) - \tfrac{1}{2}\int_0^t ds\, \Delta\phi(B_s) + \tfrac{1}{2}\int_0^t d\ell_s\, \partial_n\phi(B_s),$$

*is a continuous $\mathcal{F}_t$-martingale under $\mathbb{P}_x$ for every $x \in \bar{D}$.*

### 3.3.3 Estimates for reflecting Brownian motion

This section is devoted to collect a few estimates on reflecting Brownian motion. Unless otherwise stated they can all be found in Hsu's PhD thesis [Hsu84].

There exists a constant $c$ such that for all $x \in \bar{D}$ and all $t \in (0,1]$,

$$\int_{\partial D} \sigma(dy)\, \bar{p}_t(x,y) \leq c/\sqrt{t}, \tag{3.28}$$

where $\sigma$ is the surface measure on $\partial D$. Moreover, there exist two positive constants $c'$

and $\beta$ such that for all $x, y \in \bar{D}$, $t \geq 1$, we have

$$|\bar{p}_t(x,y) - a_D| \leq c' \, \mathrm{e}^{-\beta t}, \tag{3.29}$$

where $a_D^{-1} := \int_D dy$ is the $d$-dimensional Lebesgue measure of $D$. We deduce from (3.28) that for any $\theta > 0$, there is a constant $c_\theta < \infty$ such that, for all $x \in \bar{D}$

$$\int_0^\infty dr \int_{\partial D} \sigma(dy) \, \mathrm{e}^{-\theta r} \, \bar{p}_r(x,y) \leq c_\theta. \tag{3.30}$$

**Lemma 3.19 (Uniform moment-estimates).** *For all $n \in \mathbb{N}$, there exists $K_n > 0$ such that for all $t \geq 0$,*

$$\sup_{x \in \bar{D}} \mathbb{E}_x[(\ell_t)^n] \leq K_n \, (t^{n/2} + t^n). \tag{3.31}$$

*Proof.* From (3.28) and (3.29) we get there exists a constant $K$ such that for all $t \geq 0$, we have

$$\sup_{x \in \bar{D}} \mathbb{E}_x[\ell_t] \leq K \, (\sqrt{t} + t).$$

Notice that $\ell_t^n = n \int_0^t d\ell_s \, \ell_s^{n-1} = \int_0^t d\ell_s^{n-1} \, \ell_t = \int_0^t d\ell_s \, \ell_s^{n-1} + \int_0^t d\ell_s^{n-1} \, \ell_s$, the first two follow by direct integration, the last by integration by parts. Therefore,

$$\ell_t^n = n \int_0^t d\ell_s^{n-1} \, (\ell_t - \ell_s).$$

By the additivity property of the continuous additive functional $\ell$ we get,

$$\mathbb{E}_x\big[\ell_t^n\big] = n\mathbb{E}_x\Big[\int_0^t d\ell_s^{n-1} \, \mathbb{E}_{B_s}\big[\ell_{t-s}\big]\Big] \leq \sup_{x \in \bar{D}} \mathbb{E}_x\big[\ell_t\big] \, n\mathbb{E}_x\Big[\int_0^t d\ell_s^{n-1}\Big].$$

Hence the result follows from induction. $\square$

By [Hsu84, Theorem 2.5], the reflecting Brownian motion in $D$ has the same modulus of continuity as a standard Brownian motion in $\mathbb{R}^d$. In particular, for $T > 0$, there exists a constant $K > 0$, such that for all $t \in [0, T]$, $x \in \bar{D}$, $a \geq 0$,

$$\mathbb{P}_x\Big(\sup_{0 \leq s \leq t} |B_s - x| \geq a\Big) \leq P_x\Big(\sup_{0 \leq s \leq t} |W_s - x| \geq a/K\Big), \tag{3.32}$$

where $W = (W_t, t \geq 0)$ is under $P_x$ a standard Brownian motion in $\mathbb{R}^d$ started at $x$. From (3.32) we obtain for every $a \geq 0$,

$$\lim_{t \downarrow 0} \mathbb{P}_x\Big(\sup_{0 \leq s \leq t} |B_s - x| \geq a\Big) = 0. \tag{3.33}$$

### 3.3.4 Excursion theory for reflecting Brownian motion

Let $D$ be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with $\mathcal{C}^3$-boundary $\partial D$. Let $F_1$ and $F_2$ two relatively open subsets of $\partial D$. We assume that $F_1$ and $F_2$ are non empty, disjoint and that $\bar{F}_1 \cup \bar{F}_2 = \partial D$. We also assume that the relative boundary of $F_1$ is equal to the relative boundary of $F_2$, and that it is either empty or a $\mathcal{C}^2$-manifold of codimension 2. We denote it by $\partial F$.

In this section we want to study excursions of the reflecting Brownian motion in $D$ from the set $F_1$. In particular, we give the last exit decomposition of the reflecting Brownian motion in $D$ with respect to $F_1$. Moreover, we give a few other examples for later needs.

Consider $B = (B_t, t \geq 0)$ a reflecting Brownian motion in $D$, which is according to the last section, a continuous strong Markov process. Hence, we can apply the results of Section 3.1.2 to study excursions of $B$ out of the set $C = F_1$. In particular, there is a continuous additive functional $L = (L_t, t \geq 0)$ of $B$, called the *capacitary local time* on $F_1$, such that

$$\mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-t}\, dL_t\Big] = \mathbb{E}_x\big[\mathrm{e}^{-\tau_1}\big],$$

where $\tau_1$ is the first hitting time of $F_1$ by the reflecting Brownian motion $B$. Moreover, according to Theorem 3.5 the family of *excursion measures* $H = (H^x, x \in F_1)$ exists. Let us call the tuple $(L, H)$ the *exit system* of the reflecting Brownian motion on $F_1$. Of course, replacing $F_1$ by $F_2$, we can also consider the exit system of the reflecting Brownian motion on $F_2$. We come back to this remark in Lemma 3.20.

We now give particular applications of Theorem 3.5, which we use later. Let $\varphi \in \mathcal{B}_+(D)$. As $F_1$ has vanishing $d$-dimensional Lebesgue measure, it follows immediately from equation (3.1) that

$$\mathbb{E}_x[\varphi(B_t)] = Q_t\varphi(x) + \mathbb{E}_x\Big[\int_0^\infty \mathbf{1}_{\{s<t\}} H^{B_s}[\varphi(\omega_{t-s})]\, dL_s\Big], \tag{3.34}$$

where $(Q_t, t \geq 0)$ denotes the semigroup of the reflecting Brownian motion killed on $F_1$, which is given by

$$Q_t\varphi(x) := \mathbb{E}_x\big[\mathbf{1}_{\{t<\tau_1\}}\varphi(B_t)\big].$$

We refer to equation (3.34) as the *last exit decomposition* of the reflecting Brownian motion in $D$ with respect to $F_1$. We need it for the construction of the catalytic super-Brownian motion in $D$ with reflecting boundary in Section 3.4

Moreover, let us consider the following examples needed in Chapter 4:

1° *Example.* Let $\tau_2$ denote the first hitting time of $F_2$. For $\theta \geq 0$, let us set

$$\psi(\omega) = \mathrm{e}^{-\theta\tau_2}\,\varphi(\omega_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}} \quad \text{and} \quad Z_s(\omega) = \mathrm{e}^{-\theta s}\,\mathbf{1}_{\{\tau_2>s\}}.$$

From Theorem 3.5, we have

$$\begin{aligned}\mathbb{E}_x\Big[\int_0^{\tau_2} \mathrm{e}^{-\theta s}\,H^{B_s}\big[\mathbf{1}_{\{\tau_2<\infty\}}\,\mathrm{e}^{-\theta\tau_2}\,\varphi(\omega_{\tau_2})\big]\,dL_s\Big] &= \mathbb{E}_x\Big[\sum_{s\in G}\mathrm{e}^{-\theta s}\,\mathbf{1}_{\{\tau_2>s\}}\big[\mathrm{e}^{-\theta\tau_2}\,\varphi(\omega_{\tau_2})\big]\circ i_s\Big] \\ &= \mathbb{E}_x[\mathbf{1}_{\{\tau_2>\tau_1\}}\,\mathrm{e}^{-\theta\tau_2}\,\varphi(B_{\tau_2})], \qquad (3.36)\end{aligned}$$

since $\tau_2\circ i_s + s = \tau_2$ on $\{\tau_2 > s\}$. In particular, with $\theta = 0$, we get,

$$\mathbb{E}_x\Big[\int_0^{\tau_2} H^{B_s}[\mathbf{1}_{\{\tau_2<\infty\}}\varphi(\omega_{\tau_2})]\,dL_s\Big] = \mathbb{E}_x[\mathbf{1}_{\{\tau_2>\tau_1\}}\varphi(B_{\tau_2})]. \qquad (3.37)$$

2° *Example.* With $\psi$ defined by

$$\psi(\omega) = \int_0^\infty d\ell_t\;\mathrm{e}^{-\theta t}\,\varphi(\omega_t),$$

(recall that we denote by $\ell$ the local time on $\partial D$) and $Z_s := \mathrm{e}^{-\theta s}$ we obtain

$$\mathbb{E}_x\Big[\int_0^\infty dL_s\;\mathrm{e}^{-\theta s}\,H^{B_s}\Big[\int_0^\infty d\ell_t\;\mathrm{e}^{-\theta t}\,\varphi(\omega_t)\Big]\Big] = \mathbb{E}_x\Big[\int_{\tau_1}^\infty d\ell_t^2\;\mathrm{e}^{-\theta t}\,\varphi(B_t)\Big]. \qquad (3.38)$$

## 3.4 Catalytic super-Brownian motion in a bounded domain with reflecting boundary

Let us keep the same notation as in Section 3.3.4. In this section we *construct* a catalytic super-Brownian motion in $D$ which is catalyzed by the set $F_1$ with underlying motion a reflected Brownian motion $B$, i.e. generic particles of this superprocess move in $D$ according to a reflecting Brownian motion, when they hit $F_2$ they are just reflected and when they hit $F_1$ they may branch or die. In particular, we shall do this construction in analogy to the construction of the $\sigma$-catalytic super-Brownian motion in $\mathbb{R}^d$ in Section 3.2.

Let $B = (B_t, t \geq 0)$ be a reflecting Brownian motion in $D$, with normal reflection, started at $x \in \bar{D}$ under $\mathbb{P}_x$. Let $(\mathcal{F}_t, t \geq 0)$ be the filtration generated by $B$ completed in the usual way. See Section 3.3 for more details on $B$. For $t > 0$ we denote by $\bar{p}_t(x, y)$

the density transition kernel of $B$ and by $\ell = (\ell_t, t \geq 0)$ the local time on the boundary $\partial D$ of $B$. Moreover, let $\ell^1 = (\ell_t^1, t \geq 0)$ be the local time on $F_1$ defined by

$$d\ell_s^1 = \mathbf{1}_{F_1}(B_s)\, d\ell_s.$$

Notice that this corresponds to $\partial D$ replaced by $F_1$ in (3.27) and that the Revuz-measure associated to $\ell^1$ is given by the restriction of $\sigma$ on $F_1$, i.e. by

$$\sigma_1(dx) := \mathbf{1}_{F_1}(x)\sigma(dx).$$

Let $L = (L_t, t \geq 0)$ be the capacitary local time on $F_1$, as it was introduced in Section 3.3.4, and denote by $\mu$ the Revuz measure associated to $L$, which is the capacitary measure on the set $F_1$. In fact $L$ and $\ell^1$ do *not* coincide in general. However, in our setting, the next Lemma implies that $L$ is absolutely continuous with respect to $\ell^1$. To prove the result we have to use excursion theory for $B$ from the set $F_2$. Let us denote by $(\tilde{L}, \tilde{H})$ the exit system of the reflecting Brownian motion on $F_2$.

**Lemma 3.20 (Absolute continuity of $L$ with respect to $\ell^1$).**
*There exists $\rho \in \mathcal{B}_+(\mathbb{R}^d)$, such that $\mu(dx) = \rho(x)\sigma_1(dx)$. In particular, we have almost surely for all $t \geq 0$,*

$$dL_t = \rho(B_t)\, d\ell_t^1,$$

*in particular, the density $\rho$ is given explicitly by*

$$\rho(y) = c_d \int_D dz \, \left[ \frac{\partial g^1(z,y)}{\partial n_y} + \mathbb{E}_z \Big[ \int_0^{\tau_1} d\tilde{L}_s \; \mathrm{e}^{-s}\, \tilde{H}^{B_s}[\mathrm{e}^{-\tau_K}\, \frac{\partial g^1(e_{\tau_K}, y)}{\partial n_y}] \Big] \right],$$

*where $\tau_K$ is the first hitting time of the compact set*

$$K = \{x \in \bar{D}; d(x, F_1) \leq \varepsilon, d(x, F_1) \leq d(x, F_2)\}, \quad \varepsilon > 0,$$

*and $g^1(x,y) = \int_0^\infty \mathrm{e}^{-t}\, p_t^{\partial D}(x,y)\, dt$, and $p_t^{\partial D}$ is the density of the transition kernel of the Brownian motion killed on $\partial D$.*

*Proof.* Let $\psi$ be a non-negative continuous function defined on $\partial D$, with closed support in $F_1$. We have, for $z \in D$,

$$\mathbb{E}_z[\mathrm{e}^{-\tau_1}\, \psi(B_{\tau_1})] = \mathbb{E}_z[\mathrm{e}^{-\tau_1}\, \psi(B_{\tau_1})\mathbf{1}_{\{\tau_1 < \tau_2\}}] + \mathbb{E}_z[\mathrm{e}^{-\tau_1}\, \psi(B_{\tau_1})\mathbf{1}_{\{\tau_1 > \tau_2\}}]. \qquad (3.39)$$

We treat both summands in (3.39) separately.

1° *First summand.*
Let $\tau = \tau_1 \wedge \tau_2$ be the first hitting time of $\partial D$. Since $\psi = 0$ on $F_2$,

$$\mathbb{E}_z[\mathrm{e}^{-\tau_1}\,\psi(B_{\tau_1})\mathbf{1}_{\{\tau_1<\tau_2\}}] = \mathbb{E}_z[\mathrm{e}^{-\tau}\,\psi(B_\tau)].$$

From Theorem 8.16 in [CZ95], there is a (negative) constant $c_d$ (dependent only on $d$), such that

$$\mathbb{E}_z[\mathrm{e}^{-\tau}\,\psi(B_\tau)] = c_d \int_{\partial D} \psi(y) \frac{\partial g^1(z,y)}{\partial n_y}\,\sigma(dy), \tag{3.40}$$

where $g^1(x,y) = \int_0^\infty \mathrm{e}^{-t}\,p_t^{\partial D}(x,y)\,dt$, and $p_t^{\partial D}$ is the density of the transition kernel of the Brownian motion killed on $\partial D$.

2° *Second summand.*
We use excursion theory for excursions out of $F_2$. Therefore, let $\tilde{L} = (\tilde{L}_t, t \geq 0)$ be the capacitary local time on $F_2$ and $\tilde{H} = (\tilde{H}^x, x \in F_2)$ the associated family of excursion measures (see Section 3.1.2 and Section 3.3.4 for more details). From (3.36), with $\theta = 1$ and $F_1$ replaced by $F_2$, we deduce that

$$\mathbb{E}_z[\mathrm{e}^{-\tau_1}\,\psi(B_{\tau_1})\mathbf{1}_{\{\tau_1>\tau_2\}}] = \mathbb{E}_z\Big[\int_0^{\tau_1} \mathrm{e}^{-s}\,\tilde{H}^{B_s}[\mathrm{e}^{-\tau_1}\,\psi(\omega_{\tau_1})\mathbf{1}_{\{\tau_1<\infty\}}]\,d\tilde{L}_s\Big].$$

Let $\varepsilon > 0$ and consider the compact set

$$K = \{x \in \bar{D}; d(x,F_1) \leq \varepsilon, d(x,F_1) \leq d(x,F_2)\}, \tag{3.41}$$

and $\tau_K := \inf\{t > 0, B_t \in K\}$ the first hitting time of $K$. For $x \in F_2$, we have, using the strong Markov property of $\tilde{H}^x$ with respect to $Q_t^2$, the kernel of the reflected Brownian motion killed on $F_2$, (see [Ma75], Theorem 5.1),

$$\begin{aligned}
\tilde{H}^x[\mathrm{e}^{-\tau_1}\,\psi(e_{\tau_1})\mathbf{1}_{\{\tau_1<\tau_2\}}] &= \tilde{H}^x[\mathrm{e}^{-\tau_K}\,\mathbb{E}_{e_{\tau_K}}[\mathrm{e}^{-\tau_1}\,\psi(B_{\tau_1})\mathbf{1}_{\{\tau_1<\infty\}}]] \\
&= \tilde{H}^x\Big[\mathrm{e}^{-\tau_K}\,c_d \int_{\partial D} \psi(y)\frac{\partial g^1(e_{\tau_K},y)}{\partial n_y}\,\sigma(dy)\Big] \\
&= c_d \int_{\partial D} \psi(y) \tilde{H}^x\Big[\mathrm{e}^{-\tau_K}\,\frac{\partial g^1(e_{\tau_K},y)}{\partial n_y}\Big]\,\sigma(dy),
\end{aligned}$$

where we used (3.40) for the second equality. From this last expression, (3.40) and (3.39), we deduce that there exists a measurable non-negative function $\tilde{f}$ defined on $D \times F_1$ such that for $z \in D$,

$$\mathbb{E}_z\big[\mathrm{e}^{-\tau_1}\,\psi(B_{\tau_1})\big] = \int_{F_1} \tilde{f}(z,y)\psi(y)\sigma(dy).$$

From Proposition 3.3, we deduce that $\mu$ is absolutely continuous with respect to $\sigma$ and the density is given by $\rho(y) = \int_D \tilde{f}(z,y)\, dz$, that is

$$\rho(y) = c_d \int_D dz \, \left[\frac{\partial g^1(z,y)}{\partial n_y} + \mathbb{E}_z\Big[\int_0^{\tau_1} d\tilde{L}_s \; \mathrm{e}^{-s}\, \tilde{H}^{B_s}[\mathrm{e}^{-\tau_K}\, \frac{\partial g^1(e_{\tau_K}, y)}{\partial n_y}]\Big]\right]. \qquad (3.42)$$

□

In what follows, we proceed in the same spirit as in the Sections 2.1, 2.2.2 and 3.2.

Denote by $\ell^{1,-1} = (\ell_t^{1,-1}, t \geq 0)$ the right continuous inverse of $\ell^1$ defined by

$$\ell_t^{1,-1} := \inf\{s \geq 0 \, : \, \ell_s^1 > t\}.$$

Notice that according to [SU65, Theorem 7.2] we have $\lim_{t\to\infty} \ell_t^1 = \infty$ almost surely. Hence, $\ell_t^{1,-1}$ is finite for all $t$. It follows in the same way as we deduced Lemma 2.9, that for every $t \geq 0$, the random variable $\ell_t^{1,-1}$ is a $\mathcal{F}_t$-stopping time and that $B \circ \ell_t^{1,-1} \in F_1$ almost surely for every $t > 0$. Let $E := \mathbb{R}_+ \times F_1$. For $\hat{x} = (s,x) \in E$, we define under $\mathbb{P}_x$ the $E$-valued stochastic process $\xi = (\xi_t, t \geq 0)$ started in $\hat{x} = (s,x)$ by

$$\xi_t := (\ell_t^{1,-1} + s, B \circ \ell_t^{1,-1}). \qquad (3.43)$$

and we denote by $\mathbb{P}_{\hat{x}}^{\xi}$ the law of $\xi$ starting in $\hat{x} \in E$. Moreover, we also have to consider the process $\xi$ started at time $t > 0$ in $\hat{x} \in E$ and we denote by $\mathbb{P}_{t,\hat{x}}^{\xi}$ its law. Define the filtration $(\tilde{\mathcal{F}}_t, \geq 0)$ by $\tilde{\mathcal{F}}_t := \mathcal{F}(\ell_t^{1,-1})$. Then $\xi$ is $\tilde{\mathcal{F}}_t$-adapted and we can deduce (with the same arguments as for Proposition 2.11), that the process $\xi$ is an $E$-valued right-continuous strong Markov process with respect to the filtration $\tilde{\mathcal{F}}_t$.

As $\xi$ is an $E$-valued right-continuous Markov process we can define a *non-catalytic* superprocess with spatial motion $\xi$ (see e.g. [LG99]). So, let $\mathcal{M}_f(E)$ be the set of finite measures on $E$. For $\nu \in \mathcal{M}_f(E)$ and $t \geq 0$, let $\mathbb{P}_{t,\nu}^U$ denote the law of the quadratic (non-catalytic) superprocess $U = (U_s, s \geq t)$ with spatial motion $\xi$, starting at $\nu$ at time $t$. We shall write $\mathbb{P}_\nu^U$ for $\mathbb{P}_{0,\nu}^U$. Recall that $U$ is an $\mathcal{M}_f(E)$-valued branching Markov process.

The *total occupation measure* $\Gamma_{\sigma_1}$ of the superprocess $U$ is the random measure

$$\Gamma_{\sigma_1}(dr\, dx) := \int_t^\infty ds\, U_s(dr\, dx), \qquad (3.44)$$

defined under $\mathbb{P}^U_{t,\nu}$. It again plays a key-rôle in the construction of the $F_1$-catalytic super-Brownian motion.

**Lemma 3.21.** *Let $\phi \in \mathcal{B}_+(E)$. The function $v$ defined on $E$ by*

$$\mathbb{E}^U_\nu\Big[\exp -\langle \Gamma_{\sigma_1}, \phi\rangle\Big] = \exp -\langle \nu, v\rangle, \tag{3.45}$$

*is a non-negative solution to the integral equation*

$$v(s,x) + \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\, v^2(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\, \phi(r+s, B_r)\Big], \tag{3.46}$$

*where $s \geq 0$ and $x \in F_1$.*

*Proof.* This follows in the same way as we proved Lemma 2.16. □

Notice it is *not* clear that the integral equation (3.46) has a *unique* solution.

From the previous lemma – or even straight from the definition – we can compute the first moment of $\Gamma_{\sigma_1}$:

$$\mathbb{E}^U_\nu[\langle \Gamma_{\sigma_1}, \phi\rangle] = \int \nu(ds\, dx)\, \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\, \phi(r+s, B_r)\Big]. \tag{3.47}$$

Recall that we denote by $\tau_1$ the first hitting time of $F_1$ by the reflecting Brownian motion $B$ and notice that $\tau_1$ is finite almost surely. Let $\eta \in \mathcal{M}_f(\bar{D})$ be a finite measure on $\bar{D}$. Define $\nu_\eta \in \mathcal{M}_f(\mathbb{R}_+ \times F_1)$, the hitting distribution of $(t, B_t)$ on $F_1$ starting from $\delta_0 \otimes \eta$, i.e. for any $\psi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$,

$$\langle \nu_\eta, \psi\rangle = \int \eta(dx)\, \mathbb{E}_x[\psi(\tau_1, B_{\tau_1})].$$

Moreover, denote by $(Q_t, t \geq 0)$ the semigroup of the reflected Brownian motion $B$ killed on $F_1$, i.e.

$$Q_t\varphi(x) = \mathbb{E}_x\big[\mathbf{1}_{\{t<\tau_1\}}\varphi(B_t)\big].$$

**Definition 3.22.** *For any $\eta \in \mathcal{M}_f(\bar{D})$ we define, under $\mathbb{P}^U_{\nu_\eta}$, the $\mathcal{M}_f(\bar{D})$-valued process $Z = (Z_t, t \geq 0)$ by $Z_0 := \eta$ and for $t > 0$,*

$$\langle Z_t, \varphi\rangle = \langle \eta, Q_t\varphi\rangle + \iint \Gamma_{\sigma_1}(dr, dx)\, \mathbf{1}_{\{r<t\}}\rho(x) H^x[\varphi(\omega_{t-r})], \tag{3.48}$$

*where $\varphi \in \mathcal{B}_+(\bar{D})$. We write $\mathbb{P}^Z_\eta$ for the law of $Z$ started at $\eta$.*

Let us give an intuitive understanding of the measure valued process $Z$ defined by (3.48). The measure $Z_t$ describe a cloud of infinitesimal particles at time $t$. The first summand in (3.48) corresponds to those particles which have *not* reached the catalyst, $F_1$, at time $t$ and which are distributed according to the starting measure $\eta$ at time 0. The second corresponds to the particles which have reached the catalyst before time $t$ and perform a branching process. Particles are then released from the catalyst at time $dr$ and location $dx$ according to the random measure $\rho(x)\Gamma_{\sigma_1}(dr\,dx)$, and then they perform excursions outside the catalyst. Also, notice that (3.48) can be seen as a superprocess analogue of the one particle picture exit formula (3.34) for reflecting Brownian motion.

By Lemma 3.19, we can introduce the family of finite constants $(a_T, T \geq 0)$ given by $a_T := \sup_{x\in\bar{D}} \mathbb{E}_x\left[\ell^1_T\right] < \infty$.

**Proposition 3.23 (The Laplace functional of $Z$).**
*Let $0 < t_1 \leq \ldots \leq t_n \leq T$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{B}^b_+(\bar{D})$, such that we have $2a_T \sum_{i=1}^n \|\varphi_i\|_\infty < 1$. Then,*

$$\mathbb{E}^Z_\eta\Big[\exp - \sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle\Big] = \exp - \langle \eta, w(0,\cdot)\rangle,$$

*where $(w(s,x), s\in[0,T], x\in\bar{D})$ is the unique non-negative solution of*

$$w(s,x) + \mathbb{E}_x\left[\int_0^\infty d\ell^1_r\, w^2(r+s, B_r)\right] = \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\mathbb{E}_x[\varphi_i(B_{t_i-s})]. \tag{3.49}$$

*Proof.* Using $\phi(s,x) := \sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\rho(x)H^x[\varphi_i(\omega_{t_i-s})]$, we have

$$\mathbb{E}^Z_\eta\Big[\exp - \sum_{i=1}^n \langle Z_{t_i}, \varphi_i\rangle\Big] = \exp - \Big(\sum_{i=1}^n \langle \eta, Q_{t_i}\varphi_i\rangle + \langle \nu_\eta, \tilde{w}\rangle\Big), \tag{3.50}$$

where, by Lemma 3.21, $(\tilde{w}(s,x), s\geq 0, x\in F_1)$ is a non-negative solution of

$$\tilde{w}(s,x) + \mathbb{E}_x\left[\int_0^\infty d\ell^1_r\, \tilde{w}^2(r+s, B_r)\right] = \mathbb{E}_x\left[\int_0^\infty d\ell^1_r\, \phi(r+s, B_r)\right]. \tag{3.51}$$

By Lemma 3.20, we have for all $t \geq 0$, almost surely,

$$dL_t = \rho(B_t)\, d\ell^1_t. \tag{3.52}$$

By the definition of $\phi$ and the last exit decomposition for the reflecting Brownian motion (3.34) we get

$$
\begin{aligned}
&\mathbb{E}_x\Big[\int_0^\infty d\ell_r^1\,\phi(r+s,B_r)\Big]\\
&\qquad=\mathbb{E}_x\Big[\int_0^\infty d\ell_r\sum_{i=1}^n \mathbf{1}_{\{r+s<t_i\}}\rho(B_r)H^{B_r}[\varphi_i(\omega_{t_i-s-r})]\Big]\\
&\qquad=\mathbb{E}_x\Big[\int_0^\infty dL_r\sum_{i=1}^n \mathbf{1}_{\{r+s<t_i\}}H^{B_s}[\varphi_i(\omega_{t_i-s-r})]\Big]\\
&\qquad=\sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}\big(\mathbb{E}_x[\varphi_i(B_{t_i-s})]-Q_{t_i-s}\varphi_i(x)\big).
\end{aligned}
\tag{3.53}
$$

We define for $s\geq 0$, $x\in\bar{D}$,

$$
w(s,x):=\sum_{i=1}^n \mathbf{1}_{\{s<t_i\}}Q_{t_i-s}\varphi_i(x)+\mathbb{E}_x\big[\tilde{w}(s+\tau_1,B_{\tau_1})\big].
$$

Using the strong Markov property of $B$ at time $\tau_1$, (3.51) and (3.53), it is straightforward to check that $w$ satisfies (3.49). Notice that by construction, we have

$$
\begin{aligned}
\langle\eta,w(0,\cdot)\rangle&=\sum_{i=1}^n\langle\eta,Q_{t_i}\varphi_i\rangle+\int\eta(dx)\,\mathbb{E}_x\big[\tilde{w}(\tau_1,B_{\tau_1})\big]\\
&=\sum_{i=1}^n\langle\eta,Q_{t_i}\varphi_i\rangle+\langle\nu_\eta,\tilde{w}\rangle.
\end{aligned}
$$

By (3.50), this implies the first equality of the Lemma. To prove the uniqueness, let $w_1$ and $w_2$ be non-negative solutions of equation (3.49). Then both, $w_1$ and $w_2$ are bounded by $\sum_{i=1}^n\|\varphi_i\|_\infty$. We have

$$
w_1(s,x)-w_2(s,x)=-\mathbb{E}_x\Big[\int_0^\infty d\ell_r^1\,(w_1^2(s+r,B_r)-w_2^2(s+r,B_s))\Big].
$$

Hence,

$$
\begin{aligned}
\|w_1-w_2\|_\infty&\leq\sup_{s\in[0,T],x\in\bar{D}}\mathbb{E}_x\Big[\int_0^T d\ell_r^1\,\big|w_1^2(s+r,B_r)-w_2^2(s+r,B_s)\big|\Big]\\
&\leq 2a_T\sum_{i=1}^\infty\|\varphi_i\|_\infty\,\|w_1-w_2\|_\infty.
\end{aligned}
\tag{3.54}
$$

As $2a_T\sum_{i=1}^\infty\|\varphi_i\|_\infty<1$, we get that $w_1=w_2$ and (3.49) has a unique non-negative solution. □

It follows from Proposition 3.23 that $Z$ is a time-homogeneous Markov process. Performing moment calculations and using Kolmogorovs criterion as it is done in [De96, Théorème 4.7] (for the case of Brownian motion instead of reflecting Brownian motion) one can show, following exactly the arguments given by DELMAS, that under $\mathbb{P}^Z_\eta$ the process $Z$ has a continuous version.

**Definition 3.24 ($F_1$-catalytic SBM with reflecting boundary).**
*We call the $\mathcal{M}_f(\bar{D})$-valued continuous strong Markov process $Z$ whose finite dimensional distributions are characterized by Proposition 3.23 the* $F_1$-catalytic super-Brownian motion *in $D$ with reflecting boundary.*

**Remark 3.25. (General spatial motions.)**
The most important input to proof Proposition 3.23 were the last-exit-decomposition (3.1) and Lemma 3.21. As we derived (3.1) for a general continuous Markov process setting, it is straightforward to verify that also Lemma 3.21 is true for Markov processes, the construction of a catalytic superprocess via Definition 3.22 and Proposition 3.23 extends to continuous Markov processes: Let $Y = (Y_t, t \geq 0)$ be a continuous Markov process in $\mathbb{R}^d$ having a symmetric transition density. Suppose that $\sigma$ is a measure on $\mathbb{R}^d$ such that its fine support $\mathcal{S}$ has vanishing $d$-dimensional Lebesgue measure and that it is non-polar for $Y$. Moreover, assume that there is a unique continuous additive functional $A = (A_t, t \geq 0)$ of $Y$ associated to $\sigma$ such that $a_T := \sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_T] < \infty$ for all $T > 0$. Let $\mu$ be the capacitary measure on $\mathcal{S}$ and assume that $\mu$ is absolutely continuous with respect to $\sigma$. Then (with the obvious changes) Proposition 3.23 can be deduced by exactly the same arguments.

# Chapter 4

# Solving non-linear boundary value problem using catalytic branching

*(This chapter is based on a joint work[1] with Jean-François Delmas, Paris)*

Let $D$ be a bounded domain in $\mathbb{R}^d$ with $C^3$-boundary $\partial D$. We give a probabilistic representation formula for the non-negative solution of the mixed Dirichlet non-linear Neumann boundary value problem (DNP)

$$\begin{cases} \Delta u = 0 \quad \text{in } D, \\ u = \varphi \quad \text{on } F_2, \\ \partial_n u + 2u^2 = 0 \quad \text{on } F_1, \end{cases} \tag{4.1}$$

where $(F_1, F_2)$ is a non-trivial partition of $\partial D$, $\varphi$ is a non-negative, bounded and continuous function defined on $F_2$, and $\partial_n$ denotes the outward normal derivative on the boundary of $D$.

To solve the DNP in Section 4.1, we construct the exit measure on $F_2$ of a $F_1$-catalytic super-Brownian motion in $D$. Then we prove that the log-Laplace transform of $\varphi$ integrated with respect to the exit measure of the catalytic process on $F_2$, is a non-negative weak solution of the DNP. In a second part, Section 4.2, we show that we still have a probabilistic representation formula if the Dirichlet condition on $F_2$ is replaced by a Neumann condition.

[1] See also [DV03]; submitted to *Annales de l'Institut Henri Poincaré*.

## Notation

We keep the same notation as in Section 3.4. In particular, let $D$ be a bounded domain, i.e. a connected open subset of $\mathbb{R}^d$, $d \geq 2$, with $\mathcal{C}^3$-boundary $\partial D$. Let $\mathcal{C}^p(D)$ (resp. $\mathcal{C}^p(\bar{D})$) be the set of continuous functions defined on $D$ (resp. $\bar{D}$) of class $\mathcal{C}^p$. Let $(n_x, x \in \partial D)$ be the outward unit normal vector field and $\partial_n \varphi(x) := \langle \nabla\varphi, n_x \rangle$ denote the outward unit normal derivative on $\partial D$ at $x$ of a function $\varphi \in \mathcal{C}^1(\bar{D})$. Let $F_1$ and $F_2$ two relatively open subsets of $\partial D$. We assume that $F_1$ and $F_2$ are non empty, disjoint and that $\bar{F}_1 \cup \bar{F}_2 = \partial D$. We also assume that the relative boundary of $F_1$ is equal to the relative boundary of $F_2$, and that it is either empty or a $\mathcal{C}^2$-manifold of codimension 2. We shall denote it by $\partial F$.

Let $B = (B_t, t \geq 0)$ be a reflecting Brownian motion in $D$, with normal reflection, started at $x \in \bar{D}$ under $\mathbb{P}_x$. Moreover let $(\mathcal{F}_t, t \geq 0)$ be the filtration generated by $B$ completed the usual way (see Section 3.3 for more details about $B$). We say a property holds 'almost surely' if it holds $\mathbb{P}_x$-almost surely for all $x \in \bar{D}$. For $t > 0$, let $\bar{p}_t(x, y)$ denote the transition density of $B$. Also recall that there exists a unique continuous additive functional $\ell = (\ell_t, t \geq 0)$ of $B$ called the local time on $\partial D$, such that for every $\varphi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$ and $x \in \bar{D}$,

$$\mathbb{E}_x\left[\int_0^\infty d\ell_s\, \varphi(s, B_s)\right] = \int_0^\infty ds \int_{\partial D} \sigma(dy)\, \varphi(s, y) p_s(x, y), \tag{4.2}$$

where $\sigma$ is the surface measure on $\partial D$. In other words, $\sigma$ is the Revuz-measure of the continuous additive functional $\ell$. Furthermore, let $\ell^1 = (\ell^1_t, t \geq 0)$ be the local time on $F_1$, which can be constructed explicitly by $d\ell^1_t = \mathbf{1}_{F_1}(B_t) d\ell_t$. Moreover, let $L = (L_t, t \geq 0)$ be the capacitary local time on $F_1$ (see Section 3.3.4 for a definition of $L$) and denote by $\mu$ the Revuz-measure of $L$. Recall that $L$ is absolutely continuous with respect to $\ell^1$ (see Lemma 3.20).

A key-tool in this section is the excursion theory for reflecting Brownian motion from the set $F_1$ (Section 3.3.4 provides all tools we shall use).

## 4.1 Dirichlet condition on $F_2$

In this section we give a probabilistic representation of non-negative solutions of the DNP (4.1). The section is organized as follows: in Section 4.1.1 we construct the exit measure on $F_2$ of the $F_1$-catalytic super-Brownian motion in $D$ and characterize it in terms of its Laplace functionals. Then in Section 4.1.2 we study the dual function $w$

of the exit measure in more detail and it turns out that it is a good candidate to solve the DNP. Indeed, in Section 4.1.3 we show, assuming additional smoothness, that $w$ is a *strong* solution of the DNP. In general, we cannot assume that $w$ is as smooth. However, we prove in Section 4.1.4 that $w$ is still a solution of the DNP in a *weak* sense.

### 4.1.1 The exit measure $Z^{\mathrm{Dir}}$

In this section, we define a measure $Z^{\mathrm{Dir}}$ on $\bar{F}_2$ and characterize it in terms of its Laplace functionals. According to Section 3.4, the measure $Z^{\mathrm{Dir}}$ can be seen as the exit-measure of the $F_1$-catalytic super-Brownian motion on $F_2$. Intuitively, $Z^{\mathrm{Dir}}$ describes the spatial distribution of the generic particles of a $F_1$-catalytic super-Brownian motion in $D$ killed when they first hit $F_2$. Instead of constructing $Z^{\mathrm{Dir}}$ from the $F_1$-catalytic super-Brownian motion, we give a *direct* construction based on the subordination technique.

#### Construction of the total occupation measure $\Gamma^{\mathrm{Dir}}$

Recall that $\tau_2$ denotes the first hitting time of $F_2$ by the reflected Brownian motion $B$. Consider the local time $\ell^* = (\ell^*_t, t \geq 0)$ on $F_1$ of $B$ killed on $F_2$. It is defined by

$$d\ell^*_t = \mathbf{1}_{\{t<\tau_2\}} d\ell_t.$$

Let $\ell^{*,-1}$ denote the right continuous inverse of the continuous additive functional $\ell^*$, i.e.

$$\ell^{*,-1}_t := \inf\{s \geq 0 : \ell^*_s > t\},$$

with the convention that $\inf \emptyset = +\infty$. Let $E = (\mathbb{R}_+ \times F_1) \cup \{\delta\}$, where $\delta$ is a cemetery point. We define the $E$-valued time-homogeneous Markov process $\xi = (\xi_t, t \geq 0)$ by

$$\xi_t := \begin{cases} (\ell^{*,-1}_t, B \circ \ell^{*,-1}_t) & \text{if } \ell^{*,-1}_t < \infty, \\ \delta \quad \text{otherwise}, \end{cases}$$

and denote by $\mathbb{P}^{\xi}_{t,\hat{x}}$ its law started at $\hat{x} \in E$ at time $t \geq 0$. We also write $\mathbb{P}^{\xi}_{\hat{x}}$ for $\mathbb{P}^{\xi}_{0,\hat{x}}$. For $\nu \in \mathcal{M}_f(E)$ and $t \geq 0$, let $\mathbb{P}^X_{t,\nu}$ denote the law of the quadratic (non-catalytic) superprocess $X = (X_{s'}, s' \geq t)$ with spatial motion $\xi$, starting at $\nu$ at time $t$. We shall write $\mathbb{P}^X_{\nu}$ for $\mathbb{P}^X_{0,\nu}$. Recall that $X$ is an $\mathcal{M}_f(E)$-valued Markov process. Its total occupation measure $\Gamma^{\mathrm{Dir}}$, defined under $\mathbb{P}^X_{t,\nu}$, by

$$\Gamma^{\mathrm{Dir}}(dr\,dx) := \int_t^{\infty} ds'\, X_{s'}(dr\,dx),$$

plays the key-rôle in the construction of the exit measure. Firstly, we compute its Laplace transform. This plays the same rôle as Lemma 2.2.2 in Section 3.2 and Lemma 3.21 in Section 3.4 and is proved using the same techniques.

**Lemma 4.1 (The Laplace functional of $\Gamma^{\mathrm{Dir}}$).**
*Let $\phi \in \mathcal{B}_+(E)$ be non-negative and measurable. The function $v$ defined on $E$ by*

$$\mathbb{E}_\nu^X\Big[\exp -\langle \Gamma^{\mathrm{Dir}}, \phi\rangle\Big] = \exp -\langle \nu, v\rangle, \tag{4.3}$$

*is a non-negative solution of the integral equation*

$$v(s,x) + \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\ v^2(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\ \phi(r+s, B_r)\Big], \tag{4.4}$$

*where $s \geq 0$ and $x \in F_1$. If $\phi(\cdot, x) = \tilde{\phi}(x)$ does not depend on time, we get $v(s,x) = \tilde{v}(x)$ for all $s \geq 0$, where $\tilde{v}$ is a non-negative solution on $F_1$ of*

$$\tilde{v}(x) + \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\ \tilde{v}^2(B_r)\Big] = \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\ \tilde{\phi}(B_r)\Big]. \tag{4.5}$$

*Proof of Lemma 4.1.* As a special case of the weighted occupation time formula (see e.g. [LG99], Chapter II.3) we have for all non-negative, bounded and measurable functions $\phi$ and $h$ on $(\mathbb{R}_+ \times F_1) \cup \{\delta\}$ and $\mathbb{R}_+$ respectively, with $\phi(\delta) = 0$ and such that $h$ has compact support,

$$\mathbb{E}_{t,\nu}^X\Big[\exp - \int_t^\infty ds'\, h(s')\langle X_{s'}, \phi\rangle\Big] = \exp -\langle \nu, v_t\rangle,$$

where $v$ is the unique, non-negative solution of the integral equation,

$$v_t(\hat{x}) + \mathbb{E}_{t,\hat{x}}^\xi\Big[\int_t^\infty ds'\, v_{s'}^2(\xi_{s'})\Big] = \mathbb{E}_{t,\hat{x}}^\xi\Big[\int_t^\infty ds'\, h(s')\phi(\xi_{s'})\Big].$$

for $t \geq 0$ and $\hat{x} \in E$. By substitution ($\ell_r^* = s'$), we have with $\hat{x} = (s,x) \in E$, that

$$v_t(s,x) + \mathbb{E}_{t,(s,x)}^\xi\Big[\int_{\ell_t^{*,-1}}^\infty d\ell_r^*\ v_{\ell_r^*}^2(r, B_r)\Big] = \mathbb{E}_{t,(s,x)}^\xi\Big[\int_{\ell_t^{*,-1}}^\infty d\ell_r^*\ h(\ell_r^*)\phi(r, B_r)\Big].$$

Using the time-homogeneity of $\xi$ and $B$, this last equation can be written as

$$v_t(s,x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r^*\ v_{\ell_r^*+t}^2(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r^*\ h(\ell_r^*+t)\phi(r+s, B_r)\Big]. \tag{4.6}$$

Using the time-homogeneity of the process $X$, we also get that

$$\mathbb{E}^X_{t,\nu}\Big[\exp - \int_t^\infty ds'\, h(s')\langle X_{s'}, \phi\rangle\Big] = \mathbb{E}^X_\nu\Big[\exp - \int_0^\infty ds'\, h(s'+t)\langle X_{s'}, \phi\rangle\Big].$$

In particular, the function $v^T$ defined for $t \in [0, T]$ by the equation,

$$\mathbb{E}^X_\nu\Big[\exp - \int_0^{T-t} ds' \langle X_{s'}, \phi\rangle\Big] = \exp - \langle \nu, v^T_t\rangle,$$

is the only non-negative solution of (4.6), with $h(t) = \mathbf{1}_{[0,T]}(t)$. By monotone convergence, letting $T$ tend to $+\infty$, we get that $v^T_t$ increases point-wise to a function $v$, independent of $t$, defined by (4.3), and $v$ is a non-negative solution of

$$v(s,x) + \mathbb{E}_x\Big[\int_0^\infty d\ell^*_r\, v^2(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell^*_r\, \phi(r+s, B_r)\Big].$$

Using the definition of $\ell^*$, this last integral equation can be written as (4.4) where $s \geq 0$ and $x \in F_1$. Hence, the lemma holds for any bounded, non-negative function $\phi$. By monotone convergence it also holds for any $\phi \in \mathcal{B}_+(E)$. If $\phi(\cdot, x) = \tilde{\phi}(x)$, we get from (4.6) that

$$v_t(s,x) + \mathbb{E}_x\Big[\int_0^\infty d\ell^*_r\, v^2_{\ell^*_r+t}(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell^*_r\, h(\ell^*_r + t)\tilde{\phi}(B_r)\Big]. \tag{4.7}$$

In particular $v^{(s_0)}_t$ defined by $v^{(s_0)}_t(s,x) = v_t(s_0 + s, x)$ also solves (4.7). By uniqueness, we obtain $v^{(s_0)}_t = v_t$ for any $s_0 \geq 0$. Hence, we have that the function $v_t(s,x)$ does not depend on $s$, i.e. $v_t(s,x) = \tilde{v}_t(x)$ for any $s \geq 0$. Following the arguments after (4.6), we deduce that $v$ defined by (4.3) does not depend on time and solves (4.5). □

Notice that it is not clear that the integral equations (4.4) or (4.5) have a unique solution. From the previous lemma, we can compute the first moment of $\Gamma^{\mathrm{Dir}}$:

$$\mathbb{E}^X_\nu[\langle \Gamma^{\mathrm{Dir}}, \phi\rangle] = \int \nu(ds\,dx)\, \mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\, \phi(r+s, B_r)\Big]. \tag{4.8}$$

### Definition of the exit measure $Z^{\mathrm{Dir}}$ and its dual equation

Let $\eta \in \mathcal{M}_f(\bar{D})$ be a finite measure on $\bar{D}$. Define $\nu_\eta \in \mathcal{M}_f(\mathbb{R}_+ \times F_1)$ to be the hitting distribution of $\mathbb{R}_+ \times F_1$ by $(t, B_t)$, starting from $\delta_0 \otimes \eta$ and killed on $F_2$. Recall that we denote by $\tau_i$ the first hitting time of $F_i$. For any $\psi \in \mathcal{B}_+(\mathbb{R}_+ \times \bar{D})$, we have

$$\langle \nu_\eta, \psi\rangle = \int \eta(dx)\, \mathbb{E}_x[\mathbf{1}_{\{\tau_1 < \tau_2\}}\psi(\tau_1, B_{\tau_1})]. \tag{4.9}$$

Recall the definition of the density $\rho$ from Lemma 3.20. Also recall that we denote by $(H^x, x \in F_1)$ the family of excursion measures from the set $F_1$ for the reflecting Brownian motion in $D$ (see Sections 3.1.2 and 3.3.4 for a rigorous definition). Moreover, recall from Section 3.3.4 the excursion formula

$$\mathbb{E}_x\Big[\int_0^{\tau_2} H^{B_s}[\varphi(\mathbf{1}_{\{\tau_2<\infty\}}\omega_{\tau_2})]\,dL_s\Big] = \mathbb{E}_x[\mathbf{1}_{\{\tau_1<\tau_2\}}\varphi(B_{\tau_2})]. \tag{4.10}$$

We are now well prepared to pass to the following definition:

**Definition 4.2 (The exit measure $Z^{\mathrm{Dir}}$).**
*We define, under $\mathbb{P}^X_{\nu_\eta}$, the random measure $Z^{\mathrm{Dir}}$ on $\bar{F}_2$ by: for all $\varphi \in \mathcal{B}_+(\bar{F}_2)$,*

$$\langle Z^{\mathrm{Dir}}, \varphi\rangle = \langle \eta, Q^1(\varphi)\rangle + \iint \Gamma^{\mathrm{Dir}}(dr\,dx)\,\rho(x)H^x[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}],$$

*with $Q^1(\varphi)(r,x) = \mathbb{E}_x[\varphi(B_{\tau_2})\mathbf{1}_{\{\tau_2\le\tau_1\}}]$. We call the measure $Z^{\mathrm{Dir}}$ the* exit measure *of the $F_1$-catalytic super-Brownian motion on $F_2$, and write $\mathbb{P}^Z_\eta$ for its law.*

Firstly, we check that $Z^{\mathrm{Dir}}$ is finite by performing a first moment calculation.

**Lemma 4.3 (First moment and finiteness of $Z^{\mathrm{Dir}}$).**
*For every $\eta \in \mathcal{M}_f(\bar{D})$ and every $\varphi \in \mathcal{B}_+(\bar{F}_2)$ we have for the first moment of $Z^{\mathrm{Dir}}$,*

$$\mathbb{E}^Z_\eta[\langle Z^{\mathrm{Dir}}, \varphi\rangle] = \int \eta(dx)\,\mathbb{E}_x[\varphi(B_{\tau_2})].$$

*In particular, the exit measure $Z^{\mathrm{Dir}}$ is an almost surely finite random measure on $\bar{F}_2$.*

*Proof.* Using the equation for the first moment of $\Gamma^{\mathrm{Dir}}$ (4.8), we can compute

$$\begin{aligned}
\mathbb{E}^Z_\eta[\langle Z^{\mathrm{Dir}}, \varphi\rangle] &= \langle \eta, Q^1(\varphi)\rangle + \mathbb{E}^X_{\nu_\eta}\Big[\iint \Gamma(dr,dx)\,\rho(x)H^x[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\\
&= \langle \eta, Q^1(\varphi)\rangle + \int \nu_\eta(ds,dx)\,\mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_r\,\rho(B_r)H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\\
&= \langle \eta, Q^1(\varphi)\rangle + \int \eta(dx)\,\mathbb{E}_x\Big[\mathbf{1}_{\{\tau_1<\tau_2\}}\mathbb{E}_{B_{\tau_1}}\Big[\int_0^{\tau_2} dL_r\,H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\Big]\\
&= \int \eta(dx)\,\mathbb{E}_x[\varphi(B_{\tau_2})\mathbf{1}_{\{\tau_2\le\tau_1\}}] + \int \eta(dx)\,\mathbb{E}_x\Big[\int_0^{\tau_2} dL_r\,H^{B_r}[\varphi(e_{\tau_2})\mathbf{1}_{\{\tau_2<\infty\}}]\Big]\\
&= \int \eta(dx)\,\mathbb{E}_x[\varphi(B_{\tau_2})\mathbf{1}_{\{\tau_2\le\tau_1\}}] + \int \eta(dx)\,\mathbb{E}_x\Big[\mathbf{1}_{\{\tau_1<\tau_2\}}\varphi(B_{\tau_2})\Big]\\
&= \int \eta(dx)\,\mathbb{E}_x[\varphi(B_{\tau_2})],
\end{aligned}$$

where we used Lemma 3.20 (or (3.52)) and the definition of $\nu_\eta$, (4.9), for the third equality, the strong Markov property for $B$ for the fourth and (4.10) for the fifth. ☐

Recall the definition of the constant $\gamma = \sup_{x \in \bar{D}} \mathbb{E}_x[\ell_{\tau_2}] < \infty$ from Lemma 4.35.

**Lemma 4.4 (The Laplace functional of $Z^{\mathrm{Dir}}$).**
*For any measurable and non-negative function $\varphi \in \mathcal{B}_+(\bar{F}_2)$ we have,*

$$\mathbb{E}^Z_\eta \left[ \exp - \langle Z^{\mathrm{Dir}}, \varphi \rangle \right] = \exp - \langle \eta, w \rangle,$$

*where $(w(x), x \in \bar{D})$ is a non-negative solution of the integral equation on $\bar{D}$ given by*

$$w(x) + \mathbb{E}_x \left[ \int_0^{\tau_2} d\ell_r \, w^2(B_r) \right] = \mathbb{E}_x[\varphi(B_{\tau_2})]. \tag{4.11}$$

*If we additionally assume that $2\gamma \|\varphi\|_\infty < 1$, then the non-negative solution $w$ is also unique.*

*Proof.* Using $\phi(x, r) := \rho(x) \, H^x[\varphi(e_{\tau_2})]$, we can compute

$$\begin{aligned}
\mathbb{E}^Z_\eta \left[ \exp - \langle Z^{\mathrm{Dir}}, \varphi \rangle \right] &= \mathbb{E}^X_{\nu_\eta} \left[ \exp - (\langle \eta, Q^1(\varphi) \rangle + \langle \Gamma^{\mathrm{Dir}}, \phi \rangle) \right] \\
&= \exp - (\langle \eta, Q^1(\varphi) \rangle + \langle \nu_\eta, v \rangle),
\end{aligned}$$

where, by the second part of Lemma 4.1, the function $v$ is a non-negative solution on $F_1$ of the integral equation,

$$\begin{aligned}
v(x) + \mathbb{E}_x \left[ \int_0^{\tau_2} d\ell_r \, v^2(B_r) \right] &= \mathbb{E}_x \left[ \int_0^{\tau_2} d\ell_r \, \rho(B_r) H^{B_r}[\varphi(e_{\tau_2}) \mathbf{1}_{\{\tau_2 < \infty\}}] \right] \\
&= \mathbb{E}_x \left[ \int_0^{\tau_2} dL_r \, H^{B_r}[\varphi(e_{\tau_2}) \mathbf{1}_{\{\tau_2 < \infty\}}] \right] \\
&= \mathbb{E}_x[\mathbf{1}_{\{\tau_1 < \tau_2\}} \varphi(B_{\tau_2})],
\end{aligned} \tag{4.12}$$

where we used (3.52) for the second equality and (4.10) for the last equality. We define for $x \in \bar{D}$,

$$w(x) := Q^1(\varphi)(x) + \mathbb{E}_x[\mathbf{1}_{\{\tau_1 < \tau_2\}} v(B_{\tau_1})].$$

Notice that $\langle \eta, w \rangle = \langle \eta, Q^1(\varphi) \rangle + \langle \nu_\eta, v \rangle$. In particular, we have

$$\mathbb{E}^Z_\eta \left[ \exp - \langle Z^{\mathrm{Dir}}, \varphi \rangle \right] = \exp - \langle \eta, w \rangle.$$

Using the strong Markov property of $B$ and (4.12), we get that $w$ is a non-negative solution of (4.11). The proof of uniqueness is similar to the one for Proposition 3.23. $\square$

### 4.1.2 Properties of the dual function $w$

Fix $\varphi \in \mathcal{B}_+(\bar{F}_2)$ continuous (and of course bounded). Let $w$ be the non-negative function defined on $\bar{D}$ by

$$w(x) := -\log \mathbb{E}^Z_{\delta_x}\left[\exp -\langle Z^{\mathrm{Dir}}, \varphi\rangle\right], \tag{4.13}$$

where $\delta_x$ is the Dirac mass at $x$. We refer $w$ to be the *dual function* of the exit measure $Z^{\mathrm{Dir}}$. Notice that $w$ is bounded, as (4.11) implies $\|w\|_\infty \le \|\varphi\|_\infty$. In this section, we establish some properties of the function $w$. We use techniques similar to those developed in [Ab00]. The following Lemma turns out to be crucial to study the properties of the function $w$.

**Lemma 4.5.** *Let $x \in \bar{D}$, and $T$ be a finite $\mathcal{F}_t$-stopping time. Then, we have*

$$\mathbb{E}_x[w(B_{\tau_2\wedge T})] - w(x) = \mathbb{E}_x\left[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\right].$$

*Proof.* Applying the strong Markov property at time $\tau_2 \wedge T$, the integral equation for $w$ yields,

$$\begin{aligned}
\mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_s\, w^2(B_s)\Big] &= \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x\Big[\int_{\tau_2\wedge T}^{\tau_2} d\ell_s\, w^2(B_s)\Big]\\
&= \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x\Big[\mathbb{E}_{B_{\tau_2\wedge T}}\Big[\int_0^{\tau_2} d\ell_s\, w^2(B_s)\Big]\Big]\\
&= \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x\big[\mathbb{E}_{B_{\tau_2\wedge T}}[\varphi(B_{\tau_2})] - w(B_{\tau_2\wedge T})\big]\\
&= \mathbb{E}_x\Big[\int_0^{\tau_2\wedge T} d\ell_s\, w^2(B_s)\Big] + \mathbb{E}_x[\varphi(B_{\tau_2})] - \mathbb{E}_x[w(B_{\tau_2\wedge T})].
\end{aligned}$$

On the other hand, the integral equation for $w$ also gives,

$$\mathbb{E}_x\Big[\int_0^{\tau_2} d\ell_s\, w^2(B_s)\Big] = \mathbb{E}_x[\varphi(B_{\tau_2})] - w(x),$$

which completes the proof of the lemma. □

Using Lemma 4.5, we can easily show that the function $w$ is harmonic in $D$.

**Proposition 4.6 (Harmonicity of the dual function $w$).**
*The function $w$ is harmonic in $D$, i.e. it solves Laplaces equation $\Delta u = 0$ in $D$. In particular we have that $w$ is in $\mathcal{C}^2(D)$.*

*Proof.* Let $x \in D$. As $D$ is open, we may find an open ball around $x$ denoted by $O_x$ such that $O_x \subset D$. Let $T := inf\{t > 0 : B_t \in \partial O_x\}$ be the first hitting time of the boundary, $\partial O_x$, of $O_x$. As $T < \tau_2$ almost surely, Lemma 4.5 gives that $w(x) = \mathbb{E}_x[w(B_T)]$. Hence, $w$ is harmonic in $D$ and therefore belongs to $\mathcal{C}^2(D)$. □

For $A, B \subseteq \mathbb{R}^d$ let $d(A, B) := \inf\{|a - b| : a \in A, b \in B\}$ denote the Euclidean distance between the sets $A$ and $B$.

**Proposition 4.7 (Continuity of the dual function $w$ on $\bar{D}$).**
*The function $w$ is continuous on $\bar{D}$.*

*Proof.* As we already know that $w$ is continuous in $D$, it remains to deal with $\partial D$.
*First case.* Let $y \in \bar{F}_2$. As $w$ is bounded, we have

$$\mathbb{E}_x\left[\int_0^{\tau_2} d\ell_s w^2(B_s)\right] \leq \|w\|_\infty^2 \mathbb{E}_x[\ell_{\tau_2}],$$

which converges to 0 as $x \to y$ by Lemma 4.35. As $\varphi$ is continuous, we are able to deduce using Lemma 4.37,

$$\lim_{x \to y} \mathbb{E}_x[\varphi(B_{\tau_2})] = \varphi(y).$$

Hence by (4.11) $w$ is continuous at $y$.

*Second case.* Let $y \in F_1$. As $F_1$ is relatively open there exists an open ball $O_y$ around $y$ such that $O_y \cap F_2 = \emptyset$ and $d(O_y \cap \bar{D}, F_2) > 0$. By Lemma 4.5 applied to the deterministic time $T = t > 0$, we have for all $x \in O_y \cap \bar{D}$,

$$\begin{aligned} w(x) &= \mathbb{E}_x[w(B_{\tau_2})\mathbf{1}_{\{\tau_2 \leq t\}}] + \mathbb{E}_x[w(B_t)\mathbf{1}_{\{\tau_2 > t\}}] - \mathbb{E}_x\Big[\int_0^{\tau_2 \wedge t} d\ell_s w^2(B_s)\Big] \\ &= \mathbb{E}_x[w(B_{\tau_2})\mathbf{1}_{\{\tau_2 \leq t\}}] + \mathbb{E}_x[w(B_t)] - \mathbb{E}_x[w(B_t)\mathbf{1}_{\{\tau_2 \leq t\}}] - \mathbb{E}_x\Big[\int_0^{\tau_2 \wedge t} d\ell_s w^2(B_s)\Big]. \end{aligned}$$

Now, for a fixed $t > 0$, the function $x \mapsto \mathbb{E}_x[w(B_t)]$ is continuous, and all other expressions in the right-hand-side of the last equation converge to zero, uniformly in $x \in O_y \cap \bar{D}$, as $t \downarrow 0$ using Lemma 4.40 and Lemma 3.19, with $n = 1$, for the last term. □

The following corollary is now immediate from Lemma 4.5 and Proposition 4.7:

**Corollary 4.8 (The martingale associated to the dual function $w$).**
*The stochastic process $M^{\mathrm{Dir}} = (M_t^{\mathrm{Dir}}, t \geq 0)$ defined by*

$$M_t^{\mathrm{Dir}} := w(B_{t \wedge \tau_2}) - w(B_0) - \int_0^{t \wedge \tau_2} d\ell_s\, w^2(B_s)$$

*is a continuous $\mathcal{F}_t$-martingale.*

### 4.1.3 Strong solutions of the Neumann Problem

In this section we use the results obtained in Section 4.1.2 that assuming additional smoothness of the function $w$ defined by (4.13), it is a unique strong solution of the following mixed Dirichlet non-linear Neumann boundary value problem:

**Definition 4.9 (Strong solution of the DNP).**
*A function $u \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ which is harmonic in $D$ and satisfies*

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = \varphi & \text{on } F_2, \\ \partial_n u + 2\,u^2 = 0 & \text{on } F_1. \end{cases} \tag{4.14}$$

*is called a* strong solution *of the mixed Dirichlet non-linear Neumann boundary value problem (DNP).*

**Proposition 4.10 (Uniqueness of strong solutions).** *If $u$ and $v$ are both strong solutions of the DNP (4.14) then $u = v$.*

*Proof.* Let $u$ and $v$ be both strong solutions and define $w := u - v$. Then $\Delta w = 0$ in $D$ and $w = 0$ on $F_2$. Moreover,

$$2(u(y)^2 - v(y)^2) = -\partial_n(u - v)(y) \tag{4.15}$$

on $F_1$. As $w$ is harmonic we obtain by Green's first identity,

$$0 = \int_D dx\, w(x)\,\Delta w(x) = \int_{\partial D} \sigma(dy)\, w(y)\partial_n w(y) - \int_D dx\,|\nabla w(x)|^2. \tag{4.16}$$

Therefore, using $w = 0$ on $F_2$, the definition of $w$, (4.15) and the binomial formula

$$-2\int_{F_1} \sigma(dy)\,(u(y) - v(y))^2(u(y) + v(y)) = \int_D dx\,|\nabla w(x)|^2 \geq 0. \tag{4.17}$$

As $u$ and $v$ are both nonnegative, the integrand on the left hand side is nonnegative. Hence, $u = v$ almost everywhere on $F_1$ and by continuity $u = v$ on $\bar{F}_1$. Therefore we obtain $u = v$ on $\partial D$. As $u$ and $v$ are harmonic we deduce $u = v$ on $D$. □

**Proposition 4.11 (Representation of strong solutions).**
*Assume that the function $w$ defined by (4.13) is in $\mathcal{C}^1(\bar{D})$ then $w$ is the unique strong solution of the DNP (4.14).*

*Proof.* By the Propositions 4.6 and 4.7, we have $w \in \mathcal{C}^2(D) \cap \mathcal{C}(\bar{D})$ and $\Delta w = 0$ in $D$. Moreover, (4.11) implies $w = \varphi$ on $F_2$. Assume additionally that $w \in \mathcal{C}^1(\bar{D})$. Then it remains to show that

$$\partial_n w + 2w^2 = 0. \tag{4.18}$$

on $F_1$. Let $x \in \bar{D}$ and $T$ a bounded $\mathcal{F}_t$-stopping time such that $T \leq \tau_2$ almost surely. As $w \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ Lemma 3.18 implies that the process $Y = (Y_t; t \geq 0)$ defined by

$$Y_t := w(B_t) - w(B_0) - \tfrac{1}{2} \int_0^t ds\, \Delta w(B_s) + \tfrac{1}{2} \int_0^t d\ell_s \partial_n w(B_s)$$

is an $\mathcal{F}_t$-martingale. Hence, since $\Delta w = 0$ on $D$ and $0 = \mathbb{E}_x[Y_0] = \mathbb{E}_x[Y_t]$, we have

$$\mathbb{E}_x[w(B_T)] - w(x) + \tfrac{1}{2}\mathbb{E}_x\Big[\int_0^T d\ell_r\, \partial_n w(B_r)\Big] = 0.$$

On the other hand, it is immediate from Lemma 4.5 that

$$\mathbb{E}_x[w(B_T)] - w(x) = \mathbb{E}_x\Big[\int_0^T d\ell_r\, w^2(B_r)\Big].$$

Hence, we also have

$$\mathbb{E}_x\Big[\int_0^T d\ell_r\, \big(\tfrac{1}{2}\partial_n w(B_r) + w^2(B_r)\big)\Big] = 0. \tag{4.19}$$

Let $x \in F_1$ and suppose that

$$\partial_n w(x) + 2w^2(x) > 0.$$

As we assumed $w \in \mathcal{C}^1(\bar{D})$, there is a $\varepsilon > 0$ such that $\partial_n w(y) + 2w^2(y) > 0$ for all $y \in O_\varepsilon(x) \cap F_1$, where $O_\varepsilon(x)$ denotes an open ball around $x$ of radius $\varepsilon$. Define the $\mathcal{F}_t$-stopping time

$$T := \inf\{t > 0 \,:\, B_t \in F_1 \text{ and } \partial_n w(B_t) + 2w^2(B_t) \leq 0\} \wedge \tau_2 \wedge 1.$$

Then $T$ is bounded and $T \leq \tau_2$ almost surely. Moreover, by the continuity of the reflecting Brownian motion $T > 0$ almost surely with respect to $\mathbb{P}_x$ and therefore $\mathbb{P}_x(\ell_T > 0) > 0$, leading to

$$\mathbb{E}_x\Big[\int_0^T d\ell_r\, \big(\tfrac{1}{2}\partial_n w(B_r) + w^2(B_r)\big)\Big] > 0,$$

which is in contradiction to equation (4.19). $\square$

### 4.1.4 Weak solutions of the Neumann Problem

In general, it is not clear that $w$ belongs to $\mathcal{C}^1(\bar{D})$. In other words, it is not clear that a strong solution exists in general. However, we still obtain that $w$ is a solution (4.14) in a *weak* sense. Nevertheless, it is not clear that weak solutions of the DNP (4.14) are still unique. Let us define a set of test functions by

$$S_1 := \Big\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D});\ \Delta\phi \text{ is bounded in } D,\ \partial_n\phi = 0 \text{ on } F_1, \phi = 0 \text{ on } F_2\Big\},$$

and recall that we assume $\varphi \in \mathcal{C}_+(\bar{F}_2)$ to be a continuous function.

**Definition 4.12 (Weak solution of the DNP).**
*A bounded function $u \in \mathcal{B}_+(\bar{D})$ is called a weak solution of the mixed Dirichlet non-linear Neumann boundary value problem DNP (4.14) if $u \in \mathcal{C}(\bar{D})$ and for every test function $\phi \in S_1$,*

$$\int_D dx\, u(x)\, \Delta\phi(x) = \int_{F_2} \sigma(dy)\, \partial_n\phi(y)\varphi(y) + 2\int_{F_1} \sigma(dy)\, \phi(y)u^2(y). \tag{4.21}$$

If $u \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D})$ it follows directly by Greens second identity,

$$\int_D dx\, \big[u(x)\, \Delta\phi(x) - \Delta u(x)\, \phi(x)\big] = \int_{\partial D} \sigma(dy)\, \big[u(x)\, \partial_n\phi(x) - \partial_n u(x)\, \phi(x)\big]$$

that any strong solution is also a weak solution of the DNP (4.14). This indeed motivates Definition 4.12. We now turn our attention to a sufficient condition for a function $u$ being a weak solution of the DNP in terms of a martingale problem. A set of technical lemmas on reflecting Brownian motion needed for the proof are postponed to Section 4.3.

**Proposition 4.13 (A martingale condition for weak solutions).**
*A non-negative function $u \in \mathcal{C}(\bar{D})$ such that $u = \varphi$ on $F_2$, is a weak solution of the DNP (4.14), if the process $M = (M_t, t \geq 0)$ defined on $[0, +\infty)$ by*

$$M_t := u(B_{t\wedge\tau_2}) - u(B_0) - \int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)$$

*is a continuous $\mathcal{F}_t$-martingale.*

*Proof.* Assume that $u \in \mathcal{C}(\bar{D})$ is non-negative and $M = (M_t, t \geq 0)$, as defined in the statement of the proposition, is a continuous $\mathcal{F}_t$-martingale. We have for $x \in \bar{D}$,

$$\mathbb{E}_x\big[u(B_{t\wedge\tau_2})\big] - u(x) = \mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big].$$

Rewriting this equation, we obtain

$$\mathbb{E}_x\big[u(B_t)\big] - u(x) = \mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big] - \mathbb{E}_x\big[\mathbf{1}_{\{\tau_2<t\}}\big(u(B_{\tau_2}) - u(B_t)\big)\big].$$

Multiplying with $\phi \in S_1$ and integrating over $D$ yields,

$$\begin{aligned}\int_D dx\, \phi(x)\, &\big[\mathbb{E}_x\big[u(B_t)\big] - u(x)\big] \\ &= \int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big] - \int_D dx\, \phi(x)\mathbb{E}_x\big[\mathbf{1}_{\{\tau_2<t\}}\big(u(B_{\tau_2}) - u(B_t)\big)\big].\end{aligned} \tag{4.22}$$

Using the symmetry of the reflecting Brownian motion, we can rewrite the left hand side:

$$\int_D dx\, \phi(x)\mathbb{E}_x\big[u(B_t) - u(x)\big] = \int_D dx\, u(x)\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big].$$

By Lemma 3.18, the process $Y = (Y_t, t \geq 0)$ defined by

$$Y_t := \phi(B_t) - \phi(B_0) - \tfrac{1}{2}\int_0^t ds\, \Delta\phi(B_s) + \tfrac{1}{2}\int_0^t d\ell_s \partial_n\phi(B_s)$$

is a continuous $\mathcal{F}_t$-martingale. Hence, as $0 = \mathbb{E}_x[Y_0] = \mathbb{E}_x[Y_t]$, we have

$$\mathbb{E}_x\big[\phi(B_t) - \phi(x)\big] = \tfrac{1}{2}\mathbb{E}_x\Big[\int_0^t ds\, \Delta\phi(B_s)\Big] - \tfrac{1}{2}\mathbb{E}_x\Big[\int_0^t d\ell_s\, \partial_n\phi(B_s)\Big].$$

Therefore, we can rewrite (4.22) as

$$\begin{aligned}\frac{1}{t}\int_D dx\, u(x)\mathbb{E}_x&\Big[\int_0^t ds\, \Delta\phi(B_s)\Big] - \frac{1}{t}\int_D dx\, u(x)\mathbb{E}_x\Big[\int_0^t d\ell_s\, \partial_n\phi(B_s)\Big] \\ &= 2\frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^{t\wedge\tau_2} d\ell_r\, u^2(B_r)\Big] - 2\frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\big[\mathbf{1}_{\{\tau_2<t\}}\big(u(B_{\tau_2}) - u(B_t)\big)\big],\end{aligned}$$

where we also divided by $t > 0$. By the Lemmas 4.38, 4.39, 4.41 and 4.42, and letting $t \downarrow 0$, we see that

$$\int_D dx\, u(x)\, \Delta\phi(x) - \int_{\partial D} \sigma(dy)\, u(y)\partial_n\phi(y) = 2\int_{F_1} \sigma(dy)\, \phi(y)u^2(y).$$

As $u = \varphi$ on $F_2$, $\partial_n \phi = 0$ on $F_1$, we get that $u$ is a weak solution of the DNP. □

**Corollary 4.14 (Representation of weak solutions).**
*The function $w$ given by (4.13) is a non-negative weak solution of the DNP (4.14).*

*Proof.* This follows directly from Corollary 4.8 and Proposition 4.13. □

## 4.2 Neumann condition on $F_2$

In this section, we give a probabilistic representation formula for the boundary value problem (4.14), where the Dirichlet condition on $F_2$ is replaced by a Neumann condition, i.e. we solve the mixed Neumann non-linear Neumann boundary value problem (NNP)

$$\begin{cases} \Delta u = 0 \text{ in } D \\ \partial_n u - 2\varphi = 0 \text{ on } F_2 \\ \partial_n u + 2u^2 = 0 \text{ on } F_1 \end{cases} \tag{4.23}$$

for some $\varphi \in \mathcal{C}(\bar{F}_2)$. However, due to technical difficulties, we first consider the approximating problem $\theta$-NNP given by

$$\begin{cases} \Delta u = 2\theta u \text{ in } D \\ \partial_n u - 2\varphi = 0 \text{ on } F_2 \\ \partial_n u + 2u^2 = 0 \text{ on } F_1 \end{cases} \tag{4.24}$$

for $\theta > 0$. Of course, the $\theta$-NNP is an interesting problem itself and is therefore studied in its own right. In Section 4.2.1 we construct a random measure $Z_\theta^{\mathrm{Neu}}$ on $\bar{F}_2$ which is going to play the same rôle as the exit measure $Z^{\mathrm{Dir}}$ in Section 4.1. Moreover, we establish some properties of its dual function $w_\theta$ in Section 4.2.2. Then in Section 4.2.3 we prove that $w_\theta$ is a weak solution of the boundary value problem (4.24). Finally in Section 4.2.4 we obtain a random measure $Z^{\mathrm{Neu}}$ as the limit of $Z_\theta^{\mathrm{Neu}}$ as $\theta$ tend to zero and show that its dual function is a weak solution of (4.24) with $\theta = 0$. The technique of introducing *soft killing* to avoid singularities has been already used in [AD02]. However, in this paper ABRAHAM and DELMAS worked with the Brownian snake and not with the time change technique.

### 4.2.1 The Neumann-boundary measure $Z_\theta^{\mathrm{Neu}}$

In this section we construct the random measure $Z_\theta^{\mathrm{Neu}}$ on $\bar{F}_2$ and characterize it in terms of its log-Laplace equation. We refer to $Z_\theta^{\mathrm{Neu}}$ as the *Neumann-boundary measure* on $F_2$. Then we are going to collect some properties of its dual function $w_\theta$.

**Construction of the total occupation measure $\Gamma^{\text{Neu}}$**

We use the same notation as in the last sections. For $i = 1, 2$, let $\ell^i$ denote the local time of $B$ on $F_i$, i.e.

$$d\ell^i_r = \mathbf{1}_{F_i}(B_r)\, d\ell_r.$$

Let $\mathcal{N}$ be a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx\, dt$, independent of the reflecting Brownian motion $B$. Denote by $(x_i, t_i)$ the atoms of this measure and set, for $R_0 \in [0, +\infty]$ given,

$$R_t := R_0 \wedge \inf\{x_i \,:\, t_i \le t\},$$

with the convention $\inf \emptyset = +\infty$. The Markov process $R = (R_t, t \ge 0)$ is a càdlàg decreasing $\mathbb{R}_+ \cup \{\infty\}$ valued process. Moreover, for every $t \ge 0$, $\theta \ge 0$, we have

$$\mathbb{P}(R_t > \theta | R_0) = \mathbf{1}_{\{R_0 > \theta\}} \mathbb{P}(\mathcal{N}([0, \theta] \times [0, t]) = 0) = \mathbf{1}_{\{R_0 > \theta\}}\, \mathrm{e}^{-\theta t}. \tag{4.25}$$

Let $E' := \mathbb{R}_+ \times F_1 \times [0, \infty]$. We define the $E'$-valued time-homogeneous Markov process $(\zeta_t, t \ge 0)$ by

$$\zeta_t := (\ell_t^{1,-1}, B \circ \ell_t^{1,-1}, R \circ \ell_t^{1,-1})$$

and denote by $\mathbb{P}^{\zeta}_{t,\hat{x}}$ its law started at $\hat{x} \in E'$ at time $t \ge 0$. For $\nu \in \mathcal{M}_f(E')$ and $t \ge 0$, let $\mathbb{P}^{X'}_{t,\nu}$ denote the law of the quadratic (non-catalytic) superprocess $X' = (X'_{s'}, s' \ge t)$ with spatial motion $\zeta$, starting at $\nu$ at time $t$. We shall write $\mathbb{P}^{X'}_{\nu}$ for $\mathbb{P}^{X'}_{0,\nu}$. The total occupation measure $\Gamma^{\text{Neu}}$ of the superprocess $X'$ is defined under $\mathbb{P}^{X'}_{t,\nu}$ by

$$\Gamma^{\text{Neu}}(dr\, dx\, dk) := \int_t^\infty ds'\, X'_{s'}(dr\, dx\, dk).$$

**Lemma 4.15 (The Laplace functional of $\Gamma^{\text{Neu}}$).**
*Let $\theta > 0$ and $\tilde{\phi} \in \mathcal{B}_+(E')$ be of the form $\tilde{\phi}(r, x, k) = \mathbf{1}_{\{k > \theta\}} \phi(x)$, where $\phi \in \mathcal{B}_+(F_1)$ is bounded. Then the function $\tilde{v}$ defined on $E'$ by*

$$\mathbb{E}^{X'}_{\nu}\Big[\exp - \langle \Gamma^{\text{Neu}}, \tilde{\phi} \rangle\Big] = \exp - \langle \nu, \tilde{v} \rangle,$$

*is of the form $\tilde{v}(r, x, k) = \mathbf{1}_{\{k > \theta\}} v(x)$, where $v \in \mathcal{B}(F_1)$ is a non-negative solution of the integral equation on $F_1$,*

$$v(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, v^2(B_r) \mathbf{1}_{F_1}(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, \phi(B_r) \mathbf{1}_{F_1}(B_r)\Big]. \tag{4.26}$$

**Remark 4.16.** Using equation (3.30) and that $\phi$ is bounded we obtain that the function

$$x \mapsto \mathbb{E}_x\Big[\int_0^\infty d\ell_r \, \mathrm{e}^{-\theta r} \, \phi(B_r)\mathbf{1}_{F_1}(B_r)\Big]$$

is uniformly bounded on $F_1$. Therefore, $v$ is bounded. Of course, this argument fails for $\theta = 0$.

*Proof of Lemma 4.15.* Let $\tilde{\phi} \in \mathcal{B}_+(E')$ be bounded, such that $\tilde{\phi}(r,x,k) = \mathbf{1}_{\{k>\theta\}}\phi(x)$. We proceed as in the proof of Lemma 4.1. As a special case of the weighted occupation time formula (see e.g. [LG99, Chapter II.3]) we have for all functions $h \in \mathcal{B}_+(\mathbb{R}_+)$ with compact support,

$$\mathbb{E}^{X'}_{t,\nu}\Big[-\int_t^\infty ds' \, h(s')\langle X'_{s'}, \tilde{\phi}\rangle\Big] = \exp - \langle \nu, \tilde{v}_t\rangle, \tag{4.27}$$

where $\tilde{v} \in \mathcal{B}_+(\mathbb{R}_+ \times E')$ is the unique non-negative solution of the integral equation,

$$\tilde{v}_t(\hat{x}) + \mathbb{E}^{\zeta}_{t,\hat{x}}\Big[\int_t^\infty ds' \, \tilde{v}^2_{s'}(\zeta_{s'})\Big] = \mathbb{E}^{\zeta}_{t,\hat{x}}\Big[\int_t^\infty ds' \, h(s')\tilde{\phi}(\zeta_{s'})\Big].$$

Using the definition of $\zeta$ and the substitution $\ell^1_r = s'$ we obtain with $\hat{x} = (s,x,k) \in E'$,

$$\begin{aligned}\tilde{v}_t(s,x,k) &+ \mathbb{E}^{\zeta}_{t,(s,x,k)}\Big[\int_{\ell_t^{1,-1}}^\infty d\ell^1_r \, \tilde{v}^2_{\ell^1_r}(r, B_r, R_r)\Big] \\ &= \mathbb{E}^{\zeta}_{t,(s,x,k)}\Big[\int_{\ell_t^{1,-1}}^\infty d\ell^1_r \, h(\ell^1_r)\mathbf{1}_{\{R_r>\theta\}}\phi(B_r)\Big].\end{aligned} \tag{4.28}$$

Using time homogeneity for $\zeta$ and $B$, independence between $B$ and $R$, and (4.25), we have

$$\mathbb{E}^{\zeta}_{t,(s,x,k)}\Big[\int_{\ell_t^{1,-1}}^\infty d\ell^1_r \, h(\ell^1_r)\mathbf{1}_{\{R_r>\theta\}}\phi(B_r)\Big] = \mathbf{1}_{\{k>\theta\}}\mathbb{E}_x\Big[\int_0^\infty d\ell^1_r \, h(\ell^1_r + t) \, \mathrm{e}^{-\theta r} \, \phi(B_r)\Big].$$

In particular, this quantity vanishes for $k \le \theta$. Since $\tilde{v}_t$ is non-negative, we deduce from (4.28) that $\tilde{v}(r,x,k) = 0$ if $k \le \theta$. Also notice, that for $k > \theta$, the left hand side of (4.28) does not depend on $k$. In particular, $\tilde{v}^{k_0}_t$ defined by $\tilde{v}^{k_0}_t(s,x,k) = \tilde{v}_t(s,x,k \wedge k_0)$ also solves (4.28) for any $k_0 > \theta$. By uniqueness, we get that $\tilde{v}$ does not depend on $k$ on $\{k > \theta\}$. Hence, we deduce that $\tilde{v}_t(r,x,k) = \mathbf{1}_{\{k>\theta\}}\bar{v}_t(r,x)$, where $\bar{v}_t$ is the unique non-negative solution on $F_1$ of the integral equation,

$$\bar{v}_t(s,x) + \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r \; \mathrm{e}^{-\theta r} \, \bar{v}^2_{\ell^1_r + t}(r+s, B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell^1_r \, h(\ell^1_r + t) \, \mathrm{e}^{-\theta r} \, \phi(B_r)\Big].$$

We complete the proof using similar arguments as those following equation (4.7) in the proof of Lemma 4.1. $\square$

It is not clear that (4.26) has a unique solution. However, if $\|\phi\|_\infty$ is small enough (depending on $\theta > 0$), then arguing as in the end of the proof of Proposition 3.23, one can show that (4.26) has a unique solution. Moreover, Lemma 4.15 allows us to compute the first moment of $\Gamma^{\mathrm{Neu}}$: for all $\phi \in \mathcal{B}_+(F_1)$,

$$\mathbb{E}_\nu^{X'}\big[\langle \Gamma^{\mathrm{Neu}}, \tilde{\phi}\rangle\big] = \int_{E'} \nu(ds\,dx\,dk)\,\mathbf{1}_{\{k>\theta\}}\mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\phi(B_r)\mathbf{1}_{F_1}(B_r)\Big], \tag{4.29}$$

where $\tilde{\phi}(r,x,k) = \mathbf{1}_{\{k>\theta\}}\phi(x)$.

### Definition of the Neumann-boundary measure $Z_\theta^{\mathrm{Neu}}$

Let $\eta \in \mathcal{M}_f(\bar{D})$ and define $\nu_{\eta,\theta}$ to be the law of $(\tau_1, B_{\tau_1})$, killed at an independent exponential time of rate $\theta$, with $B_0$ distributed according to $\eta$:

$$\langle \nu_{\eta,\theta}, \psi\rangle = \int \eta(dx)\,\mathbb{E}_x\big[\mathrm{e}^{-\theta\tau_1}\,\psi(\tau_1, B_{\tau_1})\big]. \tag{4.30}$$

Moreover, we write $\nu_\eta = \nu_{\eta,0}$.
We write $\nu \geq \nu'$ for $\nu, \nu' \in \mathcal{M}_f(E)$ if $\langle \nu, g\rangle \geq \langle \nu', g\rangle$ for any $g \in \mathcal{B}_+(E)$. Notice that $(\nu_{\eta,\theta}, \theta \geq 0)$ is a decreasing sequence of measures.

**Remark 4.17.** Let us write $\Gamma_\theta^{\mathrm{Neu}}$ for the random measure $\Gamma^{\mathrm{Neu}}$ defined under $\mathbb{P}^{X'}_{\nu_{\eta,\theta}\otimes\delta_\infty}$. Using the Poissonian representation of superprocesses, due to the branching property (see e.g. Theorem 4.2.1 [DLG02]), one can construct the family $(\Gamma_\theta^{\mathrm{Neu}}, \theta \geq 0)$ on the same probability space in such a way, that this family is a decreasing sequence of measures. We shall use this remark later. First we pass to the following definition:

**Definition 4.18 (The Neumann-boundary measure $Z_\theta^{\mathrm{Neu}}$).**
*Let $\theta \geq 0$. We define the random measure $Z_\theta^{\mathrm{Neu}}$ on $\bar{F}_2$ by: for all $\varphi \in \mathcal{B}_+(\bar{F}_2)$,*

$$\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle = \langle \eta, Q_\theta\varphi\rangle + \int_{E'} \Gamma_\theta^{\mathrm{Neu}}(ds\,dx\,dk)\,\mathbf{1}_{\{k>\theta\}}\rho(x)H^x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(e_r)\Big],$$

*where*

$$Q_\theta\varphi(x) := \mathbb{E}_x\Big[\int_0^{\tau_1} d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(B_r)\Big].$$

*We call $Z_\theta^{\mathrm{Neu}}$ the* Neumann-boundary measure *and denote by $\mathbb{P}^Z_{\eta,\theta}$ its law.*

Recall the following excursion formula from Section 3.3.4 (see equation (3.38)),

$$\mathbb{E}_x\Big[\int_0^\infty dL_s\ \mathrm{e}^{-\theta s}\,H^{B_s}\Big[\int_0^\infty d\ell_t\ \mathrm{e}^{-\theta t}\,\varphi(\omega_t)\Big]\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_t\ \mathrm{e}^{-\theta t}\,\varphi(B_t)\Big]. \tag{4.31}$$

From now on assume that $\theta > 0$. To see that the random measure $Z_\theta^{\mathrm{Neu}}$ is finite for $\theta > 0$, we again can perform a first moment calculation.

**Lemma 4.19 (First moment and finiteness of $Z_\theta^{\mathrm{Neu}}$).**
*For every $\eta \in \mathcal{M}_f(\bar{D})$ and every $\varphi \in \mathcal{B}_+(\bar{F}_2)$ we have for the first moment of $Z_\theta^{\mathrm{Neu}}$,*

$$\mathbb{E}^Z_{\eta,\theta}\big[\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle\big] = \int \eta(dx)\,\mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big].$$

*In particular, the Neumann-boundary measure $Z^{\mathrm{Dir}}$ is an almost surely finite random measure on $\bar{F}_2$.*

*Proof.* Using (4.29), (4.30), Lemma 3.20, the exit formula (4.31), the strong Markov property of $B$ and the definition of $Q_\theta$, we can compute,

$$\begin{aligned}
&\mathbb{E}^Z_{\eta,\theta}\big[\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle\big] \\
&\quad= \langle \eta, Q_\theta\varphi\rangle + \int_{\mathbb{R}_+\times F_1} \nu_{\eta,\theta}(ds,dx) \\
&\qquad\qquad \times \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\rho(B_r)\mathbf{1}_{F_1}(B_r)H^{B_r}\Big[\int_0^\infty d\ell_{r'}\ \mathrm{e}^{-\theta r'}\,\varphi(e_{r'}))\Big]\Big] \\
&\quad= \langle \eta, Q_\theta\varphi\rangle \\
&\qquad\quad + \int \eta(dx)\,\mathbb{E}_x\Big[\mathrm{e}^{-\theta\tau_1}\,\mathbb{E}_{B_{\tau_1}}\Big[\int_0^\infty dL_r\ \mathrm{e}^{-\theta r}\,H^{B_r}\Big[\int_0^\infty d\ell_{r'}\ \mathrm{e}^{-\theta r'}\,\varphi(e_{r'})\Big]\Big]\Big] \\
&\quad= \langle \eta, Q_\theta\varphi\rangle + \int \eta(dx)\,\mathbb{E}_x\Big[\mathrm{e}^{-\theta\tau_1}\,\mathbb{E}_{B_{\tau_1}}\Big[\int_{\tau_1}^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]\Big] \\
&\quad= \langle \eta, Q_\theta\varphi\rangle + \int \eta(dx)\,\mathbb{E}_x\Big[\int_{\tau_1}^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big] \\
&\quad= \int \eta(dx)\,\mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big].
\end{aligned}$$

For the in particular statement, notice that by (3.30) the first moment is finite. □

Recall from (3.30) that there is a family of constants $(c_\theta, \theta > 0)$ such that,

$$\mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\Big] \le c_\theta.$$

Of course, this argument to deduce the finiteness of the Neumann-boundary measure $Z_\theta^{\mathrm{Neu}}$ fails if $\theta = 0$, as the first moment is infinite if $\int_{F_2}\sigma(dy)\ \varphi(y) > 0$. This non-existence of first moments of $Z_0^{\mathrm{Neu}}$ is *the* technical reason why we are not able to tackle the boundary value problem 4.24 for $\theta = 0$ directly. This is why the whole approximation procedure is actually necessary.

**Lemma 4.20 (The Laplace functional of $Z_\theta^{\mathrm{Neu}}$).**
*Let $\theta > 0$. We have for all non-negative and measurable functions $\varphi \in \mathcal{B}_+(\bar{F}_2)$,*

$$\mathbb{E}^Z_{\eta,\theta}\big[\exp -\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle\big] = \exp -\langle \eta, w_\theta\rangle,$$

*where $(w_\theta(x), x \in \bar{D})$ is a non-negative solution of the integral equation on $\bar{D}$,*

$$w_\theta(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]. \quad (4.32)$$

*If additionally we assume that $\varphi$ is bounded with $2c_\theta^2\|\varphi\|_\infty < 1$, then the integral equation (4.32) has a unique solution.*

*Proof.* Let $\tilde{\phi} \in \mathcal{B}_+(E')$ defined by $\tilde{\phi}(s,x,k) = \mathbf{1}_{\{k>\theta\}}\phi(x)$, where

$$\phi(x) = \rho(x)H^x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(e_r)\Big].$$

We have

$$\begin{aligned}\mathbb{E}^Z_{\eta,\theta}\big[\exp -\langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle\big] &= \mathbb{E}^{X'}_{\nu_{\eta,\theta}}\big[\exp -\big(\langle \eta, Q_\theta\varphi\rangle + \langle \Gamma_\theta^{\mathrm{Neu}}, \phi\rangle\big)\big]\\ &= \exp -\big(\langle \eta, Q_\theta\varphi\rangle + \langle \nu_{\eta,\theta}, \tilde{v}_\theta\rangle\big),\end{aligned}$$

where, by Lemma 4.15, $\tilde{v}_\theta(s,x) = v_\theta(x)$ is a non-negative solution on $F_1$ of

$$\begin{aligned}v_\theta(x) &+ \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, v_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\Big]\\ &= \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, \rho(B_r)\mathbf{1}_{F_1}(B_r)H^{B_r}\Big[\int_0^\infty d\ell_{r'}\ \mathrm{e}^{-\theta r'}\,\varphi(e_{r'})\Big]\Big].\\ &= \mathbb{E}_x\Big[\int_0^\infty dL_r\ \mathrm{e}^{-\theta r}\, H^{B_r}\Big[\int_0^\infty d\ell_{r'}\ \mathrm{e}^{-\theta r'}\,\varphi(e_{r'})\Big]\Big].\\ &= \mathbb{E}_x\Big[\int_{\tau_1}^\infty d\ell_r\ \mathrm{e}^{-\theta r}\,\varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big], \end{aligned} \quad (4.33)$$

where we used Lemma 3.20 for the second equality and the exit-formula (4.31) for the last. Define for $x \in \bar{D}$,

$$w_\theta(x) := Q_\theta\varphi(x) + \mathbb{E}_x\big[\mathrm{e}^{-\theta\tau_1}\, v_\theta(B_{\tau_1})\big],$$

and notice that $w_\theta = v_\theta$ on $F_1$. Moreover, we have by construction that

$$\langle \eta, w_\theta \rangle = \langle \eta, Q_\theta \varphi \rangle + \langle \nu_{\eta,\theta}, \tilde{v}_\theta \rangle.$$

Using the strong Markov property of $B$ and (4.33) one checks that $w_\theta$ solves (4.32). If $2c_\theta^2 \|\varphi\|_\infty < 1$, we get the uniqueness as in the end of the proof of Proposition 3.23. □

**Remark 4.21.** If $\varphi$ is bounded, (4.32) implies that $w_\theta$ is bounded by $c_\theta \|\varphi\|_\infty$. In general, for $\theta = 0$, i.e. without killing, the right hand side of the integral equation (4.32) is infinite. However, in Proposition 4.31 we show that the family of functions $w_\theta$ can be bounded independently from $\theta$ under the very restrictive additional assumption that $\bar{F}_1 \cap \bar{F}_2 = \emptyset$.

### 4.2.2 Properties of the dual function $w_\theta$

In this section we establish some properties of the function $w_\theta$ which prepare to show in Section 4.2.3 that $w_\theta$ is a weak solution of the boundary value problem (4.24). Let us fix a continuous non-negative function $\varphi \in \mathcal{C}(\bar{F}_2)$ for the remainder of this chapter and define the function $w_\theta$ on $\bar{D}$ by

$$w_\theta(x) := -\log \mathbb{E}^Z_{\delta_x,\theta}\big[\exp - \langle Z_\theta^{\mathrm{Neu}}, \varphi \rangle\big].$$

We assume throughout this section that $\theta > 0$. By Remark 4.21, we have that $w_\theta$ is bounded. The following Lemma plays the same rôle in this section as Lemma 4.5 in Section 4.1.2 and can be proved using the same techniques.

**Lemma 4.22.** *Let $\theta > 0$ and let $T$ be a finite $\mathcal{F}_t$-stopping time. Then we have,*

$$\begin{aligned} w_\theta(x) &+ \mathbb{E}_x\Big[\int_0^T d\ell_r\ \mathrm{e}^{-\theta r}\, w_\theta^2(B_r) \mathbf{1}_{F_1}(B_r)\Big] \\ &= \mathbb{E}_x\Big[\mathrm{e}^{-\theta T}\, w_\theta(B_T)\Big] + \mathbb{E}_x\Big[\int_0^T d\ell_r\ \mathrm{e}^{-\theta r}\, \varphi(B_r) \mathbf{1}_{F_2}(B_r)\Big]. \end{aligned}$$

*Proof.* Let $T$ be a $\mathcal{F}_t$-stopping time. By the strong Markov property of $B$ at time $T$,

$$\begin{aligned} &\mathbb{E}_x\Big[\int_0^\infty d\ell^1_r\, e^{-\theta r} w_\theta^2(B_r)\Big] \\ &= \mathbb{E}_x\Big[\int_0^T d\ell^1_r\, e^{-\theta r} w_\theta^2(B_r)\Big] + \mathbb{E}_x\Big[\int_T^\infty d\ell^1_r\, e^{-\theta r} w_\theta^2(B_r)\Big] \\ &= \mathbb{E}_x\Big[\int_0^T d\ell^1_r\, e^{-\theta r} w_\theta^2(B_r)\Big] + \mathbb{E}_x\Big[e^{-\theta T} \mathbb{E}_{B_T}\Big[\int_0^\infty d\ell^1_r\, e^{-\theta r} w_\theta^2(B_r)\Big]\Big]. \end{aligned}$$

Using the integral equation for $w_\theta$, almost surely

$$\mathbb{E}_{B_T}\Big[\int_0^\infty d\ell_r^1\, e^{-\theta r} w_\theta^2(B_r)\Big] = \mathbb{E}_{B_T}\Big[\int_0^\infty d\ell_r^2\, e^{-\theta r}\varphi(B_r)\Big] - w_\theta(B_T).$$

Hence, using the strong Markov property again yields,

$$\mathbb{E}_x\Big[\int_0^\infty d\ell_r^1\, e^{-\theta r} w_\theta^2(B_r)\Big]$$
$$= \mathbb{E}_x\Big[\int_0^T d\ell_r^1\, e^{-\theta r} w_\theta^2(B_r)\Big] + \mathbb{E}_x\Big[\int_T^\infty d\ell_r^2\, e^{-\theta r}\varphi(B_r)\Big] - \mathbb{E}_x[e^{-\theta T} w_\theta(B_T)]. \tag{4.35}$$

On the other hand, we have

$$\mathbb{E}_x\Big[\int_0^\infty d\ell_r^1\, e^{-\theta r} w_\theta^2(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r^2\, e^{-\theta r}\varphi(B_r)\Big] - w_\theta(x). \tag{4.36}$$

Comparing (4.35) and (4.36) completes the proof. □

The next proposition shows that $w_\theta$ is 'harmonic' with respect to the operator $Gu = \frac{1}{2}\Delta u - \theta u$, i.e. we have $G w_\theta = 0$ in $D$. Notice that $G$ is the generator of Brownian motion killed at rate $\theta$.

**Proposition 4.23 (Harmonicity of $w_\theta$ with respect $G$).**
*Let $\theta > 0$. The function $w_\theta$ belongs to $\mathcal{C}^2(D)$ and solves $\Delta w_\theta = 2\theta w_\theta$.*

*Proof.* This can be proved from Lemma 4.22, using standard results on killed Brownian motion, in the same way as Proposition 4.6 is deduced from Lemma 4.5. □

**Lemma 4.24 (Continuity of the dual function $w_\theta$ on $\bar{D}$).**
*The function $w_\theta$ is continuous on $\bar{D}$.*

*Proof.* Lemma 4.22 applied to the deterministic time $T = t > 0$, yields

$$w_\theta(x) = \mathbb{E}_x\Big[\mathrm{e}^{-\theta t}\, w_\theta(B_t)\Big] + \mathbb{E}_x\Big[\int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\,\big[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\big]\Big].$$

As $\varphi$ and $w_\theta$ are bounded, we have using Lemma 3.19, that the last term of this equality decreases to 0 as $t \downarrow 0$ uniformly in $x$. As the second term is continuous in $x$ the proof is complete. □

**Corollary 4.25 (The martingale associated to the dual function $w_\theta$).**

*The stochastic process $M^{\mathrm{Neu}} = (M_t^{\mathrm{Neu}}, t \geq 0)$ defined on $[0, \infty)$ by*

$$M_t^{\mathrm{Neu}} := \mathrm{e}^{-\theta t}\, w_\theta(B_t) - w_\theta(B_0) + \int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\right],$$

*is a continuous $\mathcal{F}_t$-martingale.*

**Remark 4.26.** As $M^{\mathrm{Neu}}$ is a continuous $\mathcal{F}_t$-martingale the process $N^{\mathrm{Neu}} = (N_t^{\mathrm{Neu}}, t \geq 0)$ defined by $N_0^{\mathrm{Neu}} = 0$ and $dN_t^{\mathrm{Neu}} = \mathrm{e}^{\theta t}\, dM_t^{\mathrm{Neu}}$ is also a continuous $\mathcal{F}_t$-martingale. Notice that applying the integration by parts formula to the semimartingales $(\mathrm{e}^{\theta t}, t \geq 0)$ and $M^{\mathrm{Neu}}$ yields,

$$N_t^{\mathrm{Neu}} = w_\theta(B_t) - w_\theta(B_0) - \theta\int_0^t dr\, w_\theta(B_r) + \int_0^t d\ell_r\ \left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - w_\theta^2(B_r)\mathbf{1}_{F_1}(B_r)\right].$$

### 4.2.3 Weak solution of the $\theta$-approximation

Let us define a space of test functions $S_2$ by

$$S_2 := \Big\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D});\ \Delta\phi \text{ bounded};\ \partial_n\phi = 0 \text{ on } \partial D\Big\}.$$

**Definition 4.27 (Weak solution of the $\theta$-approximation).**
*Let $\theta \geq 0$. A function $u \in \mathcal{B}_+(\bar{D})$ is said to be a* weak solution *of the $\theta$-NNP*

$$\begin{cases} \Delta u = 2\theta u \text{ in } D, \\ \partial_n u - 2\varphi = 0 \text{ on } F_2, \\ \partial_n u + 2u^2 = 0 \text{ on } F_1, \end{cases} \tag{4.37}$$

*if $u \in \mathcal{C}(\bar{D})$ and for all $\phi \in S_2$,*

$$\int_D dx\, u(x)\Delta\phi(x) = 2\theta\int_D dx\, u(x)\phi(x) - 2\int_{F_2}\sigma(dy)\,\phi(y)\varphi(y) + 2\int_{F_1}\sigma(dy)\,u^2(y)\phi(y).$$

By Greens's second identity any non-negative strong solution of (4.37) is a weak solution.

**Proposition 4.28 (Martingale characterization of weak solutions).**
*A non-negative function $u \in \mathcal{C}(\bar{D})$ is a weak solution of the $\theta$-NNP (4.37) if and only if the process $N = (N_t, t \geq 0)$ defined by*

$$N_t = u(B_t) - u(B_0) - \theta\int_0^t dr\, u(B_r) + \int_0^t d\ell_r\ \left[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - u^2(B_r)\mathbf{1}_{F_1}(B_r)\right],$$

*is a continuous $\mathcal{F}_t$-martingale.*

*Proof.* First assume that $u \in \mathcal{C}(\bar{D})$ is a weak solution of (4.37) and let $x \in \bar{D}$. By the Markov property of $B$, we have for $0 < s < t$,

$$\mathbb{E}[N_t|\mathcal{F}_s] = N_s + \mathbb{E}_{B_s}[N_{t-s}].$$

Thus, to prove the process $N$ is a $\mathcal{F}_t$-martingale, it is enough to check that $\mathbb{E}_x[N_t] = 0$ for all $t > 0$. Let $s > 0$. As $p_s(x, \cdot) \in S_2$ (see Section 3.3), we compute, using the integral equation for $u$ and $\phi(y) = p_s(x, y)$,

$$\begin{aligned}\frac{d}{ds}\mathbb{E}_x[u(B_s)] &= \int_D dy\, u(y)\partial_s p_s(x,y) \\ &= \int_D dy\, u(y)\tfrac{1}{2}\Delta_y p_s(x,y) \\ &= \theta \int_D dy\, u(y) p_s(x,y) - \int_{F_2} \sigma(dy)\, p_s(x,y)\varphi(y) + \int_{F_1} \sigma(dy)\, p_s(x,y) u^2(y).\end{aligned}$$

For $\varepsilon > 0$, integrating from $\varepsilon$ to $t$ gives,

$$\mathbb{E}_x[u(B_t)] - \mathbb{E}_x[u(B_\varepsilon)] = \theta \int_\varepsilon^t dr\, \mathbb{E}_x[u(B_r)] - \mathbb{E}_x\Big[\int_\varepsilon^t d\ell_r^2\, \varphi(B_r)\Big] + \mathbb{E}_x\Big[\int_\varepsilon^t d\ell^1\, u^2(B_r)\Big].$$

Hence, by continuity of $u$, we see that $\mathbb{E}_x[N_t] = 0$ as $\varepsilon \downarrow 0$.
Let $u \in \mathcal{C}(\bar{D})$ and assume now that for any $x \in \bar{D}$, the process $N$ is a continuous $\mathcal{F}_t$-martingale. As $\mathbb{E}_x[N_t] = 0$, we have

$$\mathbb{E}_x[u(B_t)] - u(x) = \theta \int_0^t dr\, \mathbb{E}_x[u(B_r)] - \mathbb{E}_x\Big[\int_0^t d\ell_r^2\, \varphi(B_r)\Big] + \mathbb{E}_x\Big[\int_0^t d\ell^1\, u^2(B_r)\Big].$$

Let $\phi \in S_2$. Multiplying the last equation by $\phi$ and integrating over $D$ yields

$$\begin{aligned}\int_D dx\, u(x)\mathbb{E}_x[\phi(B_t) - \phi(x)] &= \theta \int_D dx\, \phi(x) \int_0^t dr\, \mathbb{E}_x[u(B_r)] \\ &\quad - \int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_r\, \big[\varphi(B_r)\mathbf{1}_{F_2}(B_r) - u^2(B_r)\mathbf{1}_{F_1}(B_r)\big]\Big],\end{aligned} \tag{4.39}$$

where we used for the first term the symmetry of the reflecting Brownian motion. Since $\phi \in S_2$, by Lemma 3.18 the process $Y = (Y_t, t \geq 0)$ defined by

$$Y_t := \phi(B_t) - \phi(B_0) - \tfrac{1}{2}\int_0^t ds\, \Delta\phi(B_s)$$

is also an $\mathcal{F}_t$-martingale. Hence, we have $\mathbb{E}_x[\phi(B_t) - \phi(x)] = \frac{1}{2}\mathbb{E}_x\left[\int_0^t ds\, \Delta\phi(B_s)\right]$. Thus, dividing (4.39) by $t > 0$, gives

$$\frac{1}{2t}\int_D dx\, u(x)\mathbb{E}_x\left[\int_0^t ds\, \Delta\phi(B_s)\right]$$
$$= \frac{1}{t}\theta\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t dr\, u(B_r)\Big] - \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_r\, \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big]$$
$$+ \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_r\, u^2(B_r)\mathbf{1}_{F_1}(B_r)\Big].$$

Hence, we complete the proof applying Lemmas 4.38 and 4.39. □

The fact that we could proof an 'if and only if' statement in Proposition 4.28 now enables us to prove *uniqueness* for weak solution of the $\theta$-NNP (provided the function $\varphi$ is small enough) using the martingale convergence theorem.

**Proposition 4.29 (Representation and uniqueness of weak solutions).** *The function $w_\theta$ is a non-negative weak solution of the $\theta$-NNP (4.37). If additionally $2c_\theta^2\|\varphi\|_\infty < 1$, then the solution $w_\theta$ is also unique.*

*Proof.* It follows immediately from Remark 4.26 and Proposition 4.28 that $w_\theta$ is a weak solution of (4.37). To prove uniqueness, let $u \in \mathcal{B}_+(\bar{D})$ be a weak solution of (4.37) and assume $2c_\theta^2\|\varphi\|_\infty < 1$. By Proposition 4.28 and Remark 4.26, the process $M = (M_t, t \geq 0)$ defined by

$$M_t := \mathrm{e}^{-\theta t}\, u(B_t) - u(B_0) + \int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\, \varphi(B_r)\mathbf{1}_{F_2}(B_r) - \int_0^t d\ell_r\ \mathrm{e}^{-\theta r}\, u^2(B_r)\mathbf{1}_{F_1}(B_r),$$

is a continuous $\mathcal{F}_t$-martingale, as well as $dM_t = \mathrm{e}^{-\theta t}\, dN_t$. As $u$ and $\varphi$ are bounded and by (3.30), we have that $M$ is a uniformly integrable martingale. Hence $(M_t, t \geq 0)$ converges almost surely and in $L^1$ to a limit, say $M_\infty$, with $\mathbb{E}_x[M_\infty] = \mathbb{E}_x[M_0] = 0$. Therefore, $u$ is a non-negative solution of the integral equation,

$$u(x) + \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, u^2(B_r)\mathbf{1}_{F_1}(B_r)\Big] = \mathbb{E}_x\Big[\int_0^\infty d\ell_r\ \mathrm{e}^{-\theta r}\, \varphi(B_r)\mathbf{1}_{F_2}(B_r)\Big].$$

As $2c_\theta^2\|\varphi\|_\infty < 1$, by Lemma 4.20 $w_\theta$ is the only non-negative solution of the last displayed equation. Hence, we have $u = w_\theta$. □

### 4.2.4 The limit $\theta \downarrow 0$

Let $\varphi \in \mathcal{B}_+(F_2)$ be bounded. Observe that by Remark 4.17, one can assume that $(\Gamma_\theta^{\mathrm{Neu}}, \theta > 0)$ is an increasing sequence of measures as $\theta \downarrow 0$. Notice also that $(Q_\theta \varphi, \theta > 0)$ is also an increasing sequence of functions as $\theta \downarrow 0$. From the definition of $Z_\theta^{\mathrm{Neu}}$, we deduce that the sequence $(Z_\theta^{\mathrm{Neu}}, \theta > 0)$ is also an increasing sequence of measures as $\theta \downarrow 0$. Let $Z^{\mathrm{Neu}}$ be its limit as $\theta \downarrow 0$ and notice that $Z^{\mathrm{Neu}}$ has the same law as $Z_0^{\mathrm{Neu}}$ defined by Definition 4.18. By dominated convergence, we get that $(w_\theta, \theta > 0)$ increases to a limit, say $w$, as $\theta \downarrow 0$, defined on $\bar{D}$ by

$$w(x) := -\log \mathbb{E}_{\delta_x}^{Z}\left[\exp -\langle Z^{\mathrm{Neu}}, \varphi\rangle\right].$$

**Assumption.** From now on, we assume that $\partial F = \emptyset$, that is $\bar{F}_1 \cap \bar{F}_2 = \emptyset$ for the remainder of this section. We are only able to show that $w_\theta$ is bounded *independent* of $\theta$ under this technical assumption.

The proofs of Lemma 4.30 and Proposition 4.31 where contributed by JEAN-FRANÇOIS DELMAS.

Let $(\tilde{L}, \tilde{H})$ denote the exit system on $F_2$, i.e. $\tilde{L}$ is the capacitary local time on $F_2$ and $\tilde{H} = (\tilde{H}^x : x \in F_2)$ is the family of excursion measures from $F_2$. Recall from Lemma 3.20 that the density $\rho$ of $\mu$ with respect to $\sigma$ is given explicitly by

$$\rho(y) = c_d \int_D dz \,\left[\frac{\partial g^1(z,y)}{\partial n(y)} + \mathbb{E}_z\Big[\int_0^{\tau_1} d\tilde{L}_s\ \mathrm{e}^{-s}\, \tilde{H}^{B_s}[\mathrm{e}^{-\tau_K}\, \frac{\partial g^1(e_{\tau_K}, y)}{\partial n(y)}]\Big]\right],$$

where $g^1(x,y) = \int_0^\infty \mathrm{e}^{-t} p_t^{\partial D}(x,y)\, dt$, and $p_t^{\partial D}$ is the density of the transition kernel of the Brownian motion killed on $\partial D$.

**Lemma 4.30.** *If $\partial F = \emptyset$, then the function $\rho$ is bounded.*

*Proof.* We keep the notation of this section. Since $\bar{F}_1 \cap \bar{F}_2 = \emptyset$, we can choose $\varepsilon > 0$ small enough so that for any $(x,y) \in F_1 \times F_2$, $|x-y| \geq 3\varepsilon$. In particular $K$ defined by (3.41) is in fact equal to $\{x \in \bar{D}; d(x, F_1) \leq \varepsilon\}$.
Let $P_D$ be the Poisson kernel of the Brownian motion in $D$. There exists a positive constant $C_D$ (see e.g. [CZ95, p.144]), such that for any $(z,y) \in D \times \partial D$,

$$P_D(z,y) \leq C_D d(z, \partial D)|z-y|^{-d}. \tag{4.40}$$

As $\displaystyle\int_{\partial D} \sigma(dy)\ P_D(z,y)\psi(y) = \mathbb{E}_z[\psi(B_\tau)]$, for any $\psi \in \mathcal{B}_+(\partial D)$, we deduce from (3.40) that

$$0 \leq c_d \frac{\partial g^1(z,y)}{\partial n(y)} \leq P_D(z,y). \tag{4.41}$$

From this inequality and (4.40), we deduce easily that $c_d \int_D dz \, \frac{\partial g^1(z,y)}{\partial n(y)}$ is bounded from above by a finite constant, say $C_0$, independent of $y \in F_1$. Since by construction $d(e_{\tau_K}, \partial D) > \varepsilon$ (on $\{\tau_K < \infty\}$ under $\tilde{H}^x$), we get that for any $x \in F_2$, $y \in F_1$,

$$\begin{aligned} c_d \tilde{H}^x[\mathrm{e}^{-\tau_K} \frac{\partial g^1(e_{\tau_K}, y)}{\partial n(y)}] &\leq \tilde{H}_x[\tau_K < \infty] \sup_{\{(z,y');d(z,\partial D) \geq \varepsilon \,, y' \in F_1\}} c_d \frac{\partial g^1(z,y')}{\partial n(y')} \\ &= c\tilde{H}^x[\tau_K < \infty], \end{aligned}$$

for a finite constant $c$ independent of $x \in F_2$ and $y \in F_1$, by (4.41) and (4.40). Arguing as in the proof of Lemma 8.3 of [De96], we have that

$$\sup_{x \in F_2} \tilde{H}^x \Big[ \tau_K < \infty \Big] < \infty.$$

This implies that $c_d \tilde{H}^x[\mathrm{e}^{-\tau_K} \frac{\partial g^1(e_{\tau_K}, y)}{\partial n(y)}]$ is bounded from above for $x \in F_2$ and $y \in F_1$ say by $C_1$. In particular we have

$$\rho(y) \leq C_0 + C_1 \int_D dz \, \mathbb{E}_z \left[ \int_0^\infty \mathrm{e}^{-s} \, d\tilde{L}_s \right] = C_0 + C_1 \int_D dz \, \mathbb{E}_z \left[ \mathrm{e}^{-\tau_2} \right],$$

using the definition of $\tilde{L}$. This last inequality implies that $\rho$ is bounded. □

**Proposition 4.31 (Uniform boundedness of the function $w$).**
*The function $w$ is bounded on $\bar{D}$. More precisely, there exists a finite constant $c$ independent of $\varphi$, such that for any $x \in \bar{D}$,*

$$w(x) \leq c \left( \|\varphi\|_\infty + \sqrt{\|\varphi\|_\infty} \right).$$

*Proof.* For $\varepsilon > 0$, we set $F_1^\varepsilon = \{x \in \bar{D} : d(x, F_1) \leq \varepsilon\}$, the $\varepsilon$-neighborhood of $F_1$ in $\bar{D}$. Since $\partial F = \emptyset$, there exists $\varepsilon > 0$, such that $F_1^\varepsilon \cap F_2 = \emptyset$. Let $\tau_1^\varepsilon$ the first exit time of $F_1^\varepsilon$:

$$\tau_1^\varepsilon(e) = \inf\{s > 0 : e(s) \notin F_1^\varepsilon\},$$

for $e \in \mathcal{D}$ (recall notations from Section 3.1.2). In particular, using the strong Markov property of the exit measure $H^x$, we have that for any $x \in F_1$,

$$H^x\Big[\int_0^\infty d\ell_r\Big] = H^x\Big[\mathbf{1}_{\{\tau_1^\varepsilon<\infty\}}\int_{\tau_1^\varepsilon}^{\tau_1} d\ell_r\Big] = H^x\Big[\mathbf{1}_{\{\tau_1^\varepsilon<\infty\}}\mathbb{E}_{e(\tau_1^\varepsilon)}[\ell_{\tau_1}]\Big] \le cH^x\Big[\tau_1^\varepsilon<\infty\Big],$$

where we used Lemma 4.35 for the last inequality. Arguing as in the proof of Lemma 8.3 of [De96], we have that

$$\sup_{x\in F_1} H^x\Big[\tau_1^\varepsilon<\infty\Big] < \infty.$$

This implies that $H^x\Big[\int_0^\infty d\ell_r\Big]$ is bounded on $F_1$ say by $C_0$.
Since, by Lemma 4.30, $\rho$ is bounded by a constant, say $C_1$, we get from Definition 4.18, that for $\varphi \ge 0$,

$$0 \le \langle Z_\theta^{\mathrm{Neu}}, \varphi\rangle \le \|\varphi\|_\infty \left[\int \eta(dx)\mathbb{E}_x[\ell_{\tau_1}] + C_0 C_1 \langle \Gamma_\theta^{\mathrm{Neu}}, \mathbf{1}\rangle\right].$$

From Remark 4.17, and Lemma 4.35, we get there exists a finite constant $c$, such that

$$0 \le \langle Z^{\mathrm{Neu}}, \varphi\rangle \le c\,\|\varphi\|_\infty \left[\langle \eta, \mathbf{1}\rangle + \langle \Gamma_0^{\mathrm{Neu}}, \mathbf{1}\rangle\right].$$

It is well known that the total mass of the superprocess $X'$, $\langle \Gamma_0^{\mathrm{Neu}}, \mathbf{1}\rangle$, started at $\nu_\eta$ is distributed according the law of a stable subordinator of index 1/2 at time $\langle \nu_\eta, \mathbf{1}\rangle$ (see e.g. Corollary 2.15 and the remarks before). In particular, we deduce that

$$\mathbb{E}_\eta^Z\left[\mathrm{e}^{-\langle Z^{\mathrm{Neu}},\varphi\rangle}\right] \le \mathrm{e}^{-c\langle\eta,\mathbf{1}\rangle\left(\|\varphi\|_\infty + \sqrt{\|\varphi\|_\infty}\right)},$$

for a finite constant $c$ independent of $\varphi$ and $\eta$. Since this holds for any finite measure $\eta$, this implies the proposition. □

**Lemma 4.32 (Continuity of $w$ on $\bar{D}$).**
*The function $w$ is continuous on $\bar{D}$.*

*Proof.* As $w$ is bounded, we obtain from Lemma 4.22 applied to the deterministic time $T = t > 0$ and dominated convergence,

$$w(x) = \mathbb{E}_x[w(B_t)] + \mathbb{E}_x\Big[\int_0^t d\ell_r^2\,\varphi(B_r)\Big] - \mathbb{E}_x\Big[\int_0^t d\ell_r^1\,w^2(B_r)\Big].$$

Then, we can deduce the continuity of $w$, following the proof of Lemma 4.24. □

The following Proposition is now obvious from Proposition 4.29 and dominated convergence:

**Proposition 4.33.** *The function $w$ is a weak solution to the nonlinear Neumann boundary value problem (4.37) with $\theta = 0$.*

## 4.3 Convergence lemmas for reflecting Brownian motion

In this section we give a series of technical Lemmas on reflecting Brownian motion needed in the Sections 4.1 and 4.2. For $i = 1, 2$, recall that $\tau_i$ denotes the first hitting time of $F_i$.

**Lemma 4.34.** *For any $t > 0$, the function $x \mapsto \mathbb{P}_x(\tau_i > t)$ is upper semi-continuous in $\bar{D}$. In particular, for all $y \in \bar{F}_i$, we have*

$$\lim_{x \to y; x \in \bar{D}} \mathbb{P}_x(\tau_i > t) = 0.$$

*Proof.* Notice that $\mathbb{P}_x(\tau_i > t)$ is the non-increasing limit as $\varepsilon \downarrow 0$ of

$$\mathbb{E}_x[\mathbb{P}_{B_\varepsilon}(\tau_i > t - \varepsilon)],$$

which are continuous functions of $x \in \bar{D}$. Thus the function $x \mapsto \mathbb{P}_x(\tau_i > t)$ is upper semi continuous for $t > 0$. To conclude, notice that, since $\partial D$ and $\partial F$ are smooth, any point of $\bar{F}_i$ is regular for $F_i$, and thus $\mathbb{P}_y(\tau_i > t) = 0$ for all $y \in \bar{F}_i$. □

**Lemma 4.35.** *The functions $x \mapsto \mathbb{E}_x[\tau_i]$ and $x \mapsto \mathbb{E}_x[\ell_{\tau_i}]$ are bounded on $\bar{D}$. Moreover, we have for all $y \in \bar{F}_i$,*

$$\lim_{x \to y; x \in \bar{D}} \mathbb{E}_x[\tau_i] = 0 \quad \textit{and} \quad \lim_{x \to y; x \in \bar{D}} \mathbb{E}_x\left[\ell_{\tau_i}\right] = 0.$$

*Proof.* Since $\mathbb{P}_x(\tau_i > 1) < 1$ for all $x \in \bar{D}$, we deduce from Lemma 4.34, that $\delta := \sup_{x \in \bar{D}} \mathbb{P}_x(\tau_i > 1) < 1$. By the strong Markov property of the reflecting Brownian motion, we have for any $n \in \mathbb{N}$,

$$\mathbb{P}_x(\tau_i > n) = \mathbb{E}_x\left[\mathbf{1}_{\{\tau_i > n-1\}} \mathbb{P}_{B_{n-1}}(\tau_i > 1)\right] \leq \delta \mathbb{P}_x(\tau_i > n-1),$$

and hence, by induction $\sup_{x \in \bar{D}} \mathbb{P}_x(\tau_i > n) \leq \delta^n$. Therefore,

$$\mathbb{E}_x[\tau_i] = \int_0^\infty dt\, \mathbb{P}_x(\tau_i > t) \leq \sum_{n=0}^\infty \mathbb{P}_x(\tau_i > n) \leq \frac{1}{1-\delta} < \infty. \tag{4.42}$$

Hence, $x \mapsto \mathbb{E}_x[\tau_i]$ is bounded on $\bar{D}$. Moreover, for $y \in \bar{F}_i$, the estimate in (4.42) allows us to use dominated convergence in

$$\lim_{x \to y; x \in \bar{D}} \mathbb{E}_x[\tau_i] = \lim_{x \to y; x \in \bar{D}} \int_0^\infty dt\, \mathbb{P}_x(\tau_i > t) = \int_0^\infty dt \lim_{x \to y; x \in \bar{D}} \mathbb{P}_x(\tau_i > t),$$

and the last expression is equal to zero by Lemma 4.34.
Let us now treat the function $x \mapsto \mathbb{E}_x[\ell_{\tau_i}]$. Firstly notice that using the moment estimate from Lemma 3.19 there is a $c > 0$ such that we have for every $n \in \mathbb{N}$,

$$\sup_{x \in \bar{D}} \mathbb{E}_x\left[(\ell_{n+1})^2\right]^{1/2} \leq c(n^2 + 3n + 2)^{1/2} \leq c(n^2 + 4n + 4)^{1/2} \leq c(n+2)$$

Hence, it follows using the Cauchy-Schwarz inequality, that

$$\begin{aligned}
\mathbb{E}_x[\ell_{\tau_i}] &= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbf{1}_{\{n < \tau_i \leq n+1\}} \ell_{\tau_i}\right] \\
&\leq \sum_{n=0}^{\infty} \mathbb{E}_x\left[\mathbf{1}_{\{\tau_i > n\}} \ell_{n+1}\right] \\
&\leq \sum_{n=0}^{\infty} \mathbb{P}_x(\tau_i > n)^{1/2} \mathbb{E}_x\left[(\ell_{n+1})^2\right]^{1/2} \\
&\leq c \sum_{n=0}^{\infty} \delta^{n/2}(n+2),
\end{aligned}$$

where $c$ is a finite constant independent of $x \in \bar{D}$. Hence, the function $x \mapsto \mathbb{E}_x[\ell_{\tau_i}]$ is bounded on $\bar{D}$.
The same arguments as in the previous part of the proof, show that the function $x \mapsto \mathbb{E}_x[(\ell_{\tau_i})^2]$ is bounded. Let $\varepsilon \in (0, 1]$. Using the Cauchy-Schwarz inequality for the third line and Lemma 3.19, with $n = 2$, for the fourth, we obtain for all $x \in \bar{D}$,

$$\begin{aligned}
\mathbb{E}_x[\ell_{\tau_i}] &= \mathbb{E}_x[1_{\{\tau_i > \varepsilon\}} \ell_{\tau_i}] + \mathbb{E}_x[1_{\{\tau_i \leq \varepsilon\}} \ell_{\tau_i}] \\
&\leq \mathbb{E}_x[1_{\{\tau_i > \varepsilon\}} \ell_{\tau_i}] + \mathbb{E}_x[1_{\{\tau_i \leq \varepsilon\}} \ell_{\varepsilon}] \\
&\leq \mathbb{P}_x(\tau_i > \varepsilon)^{1/2} \mathbb{E}_x[(\ell_{\tau_i})^2]^{1/2} + \mathbb{P}_x(\tau_i \leq \varepsilon)^{1/2} \mathbb{E}_x[(\ell_{\varepsilon})^2]^{1/2} \\
&\leq c(\mathbb{P}_x(\tau_i > \varepsilon)^{1/2} + \sqrt{\varepsilon}),
\end{aligned}$$

where the constant $c$ is independent of $x$. We conclude using Lemma 4.34. □

**Lemma 4.36.** *For all $\eta > 0$ and all $y \in \bar{F}_2$ we have*

$$\lim_{x \to y;\, x \in \bar{D}} \mathbb{P}_x\big(|B_{\tau_2} - x| \geq \eta\big) = 0.$$

*Proof.* First notice, that by Markov's inequality,

$$\mathbb{P}_x\big(|B_{\tau_2} - x| \geq \eta\big) \leq \eta^{-2}\mathbb{E}_x\big[|B_{\tau_2} - x|^2\big].$$

Applying Lemma 3.18 to the function $\gamma(z) := |z - x|^2$ yields that

$$M_t := |B_t - x|^2 - d\,t + \int_0^t d\ell_r\, \partial_n \gamma(B_r),$$

is a $\mathcal{F}_t$-martingale under $\mathbb{P}_x$. Notice that $|\partial_n \gamma|$ is bounded from above by a constant independent of $x$. Hence, the optional stopping theorem applied to the stopping time $t \wedge \tau_2$ and the martingale convergence theorem imply that

$$\mathbb{E}_x\big[|B_{\tau_2} - x|^2\big] \leq C\big(\mathbb{E}_x[\tau_2] + \mathbb{E}_x\big[\ell_{\tau_2}\big]\big).$$

Hence, the assertion follows by Lemma 4.35. □

**Lemma 4.37.** *Let $y \in \bar{F}_2$ and $\varphi \in \mathcal{C}(\bar{F}_2)$, then*

$$\lim_{x \to y;\, x \in \bar{D}} \mathbb{E}_x[\varphi(B_{\tau_2})] = \varphi(y).$$

*Proof.* Let $\varepsilon > 0$ and $y \in \bar{F}_2$. As $\varphi$ is continuous on $\bar{F}_2$, there exists $\delta > 0$ such that $|\varphi(y) - \varphi(z)| < \varepsilon$ for all $z \in O_\delta(y) \cap \bar{F}_2$, where $O_\delta(y)$ is the ball of radius $\delta$ centered at $y$. Hence, we have for all $x \in O_{\delta/2}(y) \cap \bar{D}$

$$\begin{aligned}
\mathbb{E}_x\big[|\varphi(B_{\tau_2}) - \varphi(y)|\big] &= \mathbb{E}_x\big[|\varphi(B_{\tau_2}) - \varphi(y)|\mathbf{1}_{\{|B_{\tau_2} - y| < \delta\}}\big] + \mathbb{E}_x\big[|\varphi(B_{\tau_2}) - \varphi(y)|\mathbf{1}_{\{|B_{\tau_2} - y| \geq \delta\}}\big] \\
&\leq \varepsilon + 2\,\|\varphi\|_\infty\, \mathbb{P}_x\big(|B_{\tau_2} - y| \geq \delta\big) \\
&\leq \varepsilon + 2\,\|\varphi\|_\infty\, \mathbb{P}_x\big(|B_{\tau_2} - x| \geq \delta/2\big).
\end{aligned}$$

We conclude using Lemma 4.36. □

**Lemma 4.38.** *For every bounded function $\phi \in \mathcal{B}(D)$ and every bounded continuous function $\psi \in \mathcal{C}(D)$,*

$$\lim_{t \downarrow 0} \frac{1}{t} \int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t ds\, \psi(B_s)\Big] = \int_D dx\, \phi(x)\psi(x).$$

*Proof.* As $\phi$ and $\psi$ are bounded we can apply dominated convergence and it is enough to show that for any $x \in D$,

$$\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}_x\Big[\int_0^t ds\, |\psi(B_s) - \psi(x)|\Big] = 0.$$

Let $x \in D$ and $\varepsilon > 0$. As $\psi$ is continuous, there is a $\eta > 0$ such that

$$|\psi(y) - \psi(x)| < \varepsilon/2$$

for all $|y - x| < \eta$. By (3.33) there is a $\delta > 0$ such that

$$\sup_{x\in\bar{D}} \mathbb{P}_x\big(|B_s - x| \geq \eta\big) < \frac{\varepsilon}{4\|\psi\|_\infty}$$

for all $s < \delta$. Hence, we have

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}_x\Big[\int_0^t ds\, |\psi(B_s) - \psi(x)|\Big] \\
&\quad = \frac{1}{t}\mathbb{E}_x\Big[\int_0^t ds\, |\psi(B_s) - \psi(x)|\mathbf{1}_{\{|B_s - x| < \eta\}}\Big] + \frac{1}{t}\mathbb{E}_x\Big[\int_0^t ds\, |\psi(B_s) - \psi(x)|\mathbf{1}_{\{|B_s - x| \geq \eta\}}\Big] \\
&\quad \leq \varepsilon/2 + 2\|\psi\|_\infty \frac{1}{t}\int_0^t ds\, \mathbb{P}_x\big(|B_s - x| \geq \eta\big) \\
&\quad \leq \varepsilon/2 + 2\|\psi\|_\infty \frac{\varepsilon}{4\|\psi\|_\infty} = \varepsilon,
\end{aligned}$$

for all $t < \delta$, which completes the proof. □

**Lemma 4.39.** *For every $\phi \in \mathcal{C}(\bar{D})$ and every bounded $\psi \in \mathcal{B}(\partial D)$,*

$$\lim_{t\downarrow 0} \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_s\, \psi(B_s)\Big] = \int_{\partial D} \sigma(dy)\, \phi(y)\psi(y).$$

*Proof.* From (4.2), and the symmetry of the density kernel $p$, we have

$$\begin{aligned}
\frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_s \psi(B_s)\Big] &= \frac{1}{t}\int_D dx\, \phi(x)\int_0^t ds \int_{\partial D} \sigma(dy)\psi(y)p_s(x,y) \\
&= \int_{\partial D} \sigma(dy)\psi(y)\,\frac{1}{t}\int_0^t ds \int_D dx\, \phi(x)p_s(y,x).
\end{aligned}$$

Then, we get the result using arguments similar to the proof of Lemma 4.38. □

**Lemma 4.40.** *For all $T > 0$, there exist constants $c > 0, K > 0$ (depending on $T$)*

*such that for all $t \in [0,T]$, $x \in \bar{D}$ with $d(x) > 0$,*

$$\mathbb{P}_x(\tau_2 \le t) \le c\frac{\sqrt{t}}{d(x)} \exp -\Big(\frac{d(x)^2}{Kt}\Big),$$

*where $d(x) = d(x, F_2)$ denotes the distance between $x$ and $F_2$.*

*Proof.* Obviously, we have

$$\mathbb{P}_x(\tau_2 \le t) \le \mathbb{P}_x\Big(\sup_{0\le s\le t} |B_s - x| \ge d(x)\Big).$$

Denote by $W = (W^1, \dots, W^d)$ the coordinates of a $d$-dimensional standard Brownian motion $W$ under the measure $P_x$. Let $T > 0$. Then the from (3.32) there is a constant $K' > 0$ such that for all $t \in [0,T]$ and $x \in \bar{D}$ with $d(x) > 0$,

$$\mathbb{P}_x\Big(\sup_{0\le s\le t} |B_s - x| \ge d(x)\Big) \le P_x\Big(\sup_{0\le s\le t} |W_s - x| \ge d(x)/K'\Big).$$

Hence, it suffices to estimate

$$P_0\Big(\sup_{0\le s\le t} |W_s| \ge d(x)/K'\Big) \le P_0\Big(\sup_{0\le s\le t} |W_s^1| \ge \frac{d(x)}{d\cdot K'}\Big) \le c\int_{d(x)/d\cdot K'}^{\infty} dr\, \frac{1}{\sqrt{2\pi t}}\, \mathrm{e}^{-r^2/2t},$$

and the Lemma follows from the elementary estimate for any $a > 0$,

$$\int_a^\infty dr\ \mathrm{e}^{-r^2/2t} = \int_a^\infty dr\, \frac{r\,\mathrm{e}^{-r^2/2t}}{r} \le \frac{-t\,\mathrm{e}^{-r^2/2t}}{r}\Bigg]_a^\infty = \frac{t}{a}\,\mathrm{e}^{-a^2/2t},$$

where we used integration by parts to obtain the inequality. □

Recall from Section 4 that

$$S_1 = \Big\{\phi \in \mathcal{C}^2(D) \cap \mathcal{C}^1(\bar{D});\ \Delta\phi \text{ is bounded in } D,\ \partial_n\phi = 0 \text{ on } F_1, \phi = 0 \text{ on } F_2\Big\}.$$

**Lemma 4.41.** *For any $\phi \in S_1$ and every bounded $\psi \in \mathcal{B}(\partial D)$,*

$$\lim_{t\downarrow 0} \frac{1}{t}\int_D dx\, \phi(x)\mathbb{E}_x\Big[\int_0^{\tau_2\wedge t} d\ell_s\, \psi(B_s)\Big] = \int_{F_1} \sigma(dy)\, \phi(y)\psi(y).$$

*Proof.* As $\phi \in S_1$, we have in particular that $\phi \in \mathcal{C}^1(\bar{D})$ and $\phi = 0$ on $F_2$. Hence, there is a constant $K > 0$ such that $\phi(x) \le K d(x)$. Let $T > 0$. We have for $t \in [0, T]$,

$$\begin{aligned}
\Big|\frac{1}{t}\int_D dx\,\phi(x)\mathbb{E}_x\Big[\int_0^t d\ell_s\psi(B_s)\Big] - \frac{1}{t}\int_D dx\,\phi(x)\mathbb{E}_x\Big[\int_0^{\tau_2\wedge t} d\ell_s\psi(B_s)\Big]\Big| \\
&\le \int_D dx\,|\phi(x)|\frac{1}{t}\mathbb{E}_x\Big[\mathbf{1}_{\{\tau_2\le t\}}\int_{t\wedge\tau_2}^t d\ell_r\,\psi(B_r)\Big] \\
&\le K\,\|\psi\|_\infty\int_D dx\,\frac{d(x)}{t}\mathbb{E}_x\big[\mathbf{1}_{\{\tau_2\le t\}}\ell_t\big] \\
&\le K\,\|\psi\|_\infty\int_D dx\,\frac{d(x)}{t}\mathbb{P}_x(\tau_2\le t)^{1/2}\,\mathbb{E}_x[(\ell_t)^2]^{1/2} \\
&\le c\int_D dx\,\frac{d(x)}{\sqrt{t}}\mathbb{P}_x(\tau_2\le t)^{1/2}, \qquad (4.44)
\end{aligned}$$

where $c$ is a constant independent of $t \in (0, T]$, and where we used the Cauchy-Schwarz inequality third inequality and Lemma 3.19, together with the elementary estimate

$$\frac{\sqrt{t+t^2}}{t} \le \frac{\sqrt{t+tT}}{t} = \sqrt{1+T}\frac{1}{\sqrt{t}},$$

for the fourth. By Lemma 4.40, we have for all $x \in D$,

$$\lim_{t\downarrow 0}\frac{1}{\sqrt{t}}\mathbb{P}_x(\tau_2\le t)^{1/2} = 0.$$

To prepare an application of dominated convergence, notice that by Lemma 4.40 there are constants $c > 0$ and $K > 0$ independent of $t \in (0, T]$ and $x \in D$ such that,

$$\frac{d(x)^2}{t}\mathbb{P}_x\big(\tau_2\le t\big) \le c\frac{d(x)^2}{t}\frac{\sqrt{t}}{d(x)}\,\mathrm{e}^{-d(x)^2/Kt} \le c\frac{d(x)^2}{t}\sqrt{K}\frac{\sqrt{t}\sqrt{t}}{d(x)},$$

where we used the elementary estimate $\sqrt{y}\,\mathrm{e}^{-y} < 1$ for any $y \ge 0$. Hence,

$$\frac{d(x)}{\sqrt{t}}\mathbb{P}_x(\tau_2\le t)^{1/2} \le \sqrt{c\sqrt{K}}.$$

Therefore we can apply dominated convergence in (4.44) to get the result using Lemma 4.39. $\square$

**Lemma 4.42.** *For every $\phi \in S_1$ and every $\psi \in \mathcal{C}(\bar{D})$, we have*

$$\lim_{t\downarrow 0}\frac{1}{t}\int_D dx\,|\phi(x)|\,\mathbb{E}_x\Big[|\psi(B_{\tau_2}) - \psi(B_t)|\mathbf{1}_{\{\tau_2<t\}}\Big] \;=\; 0.$$

*Proof.* Let $T > 0$. Let $c$ denote a constant independent of $t \in (0, T]$, which may vary.

From Lemma 4.40, we have for all $t \in [0,T]$,

$$\int_D dx\ d(x)\mathbb{P}_x(\tau_2 < t) \le c \int_D dx\, d(x) \frac{\sqrt{t}}{d(x)} \exp -\Big(\frac{d(x)^2}{Kt}\Big) \le c\sqrt{t} \int_0^\infty dr\ \ \mathrm{e}^{-r^2/Kt} \le ct.$$

As $\phi \in S_1$, there is a constant $K' > 0$ such that $|\phi(x)| \le K'd(x)$. Hence, we have for all $t \in [0,T]$,

$$\begin{aligned}
&\frac{1}{t}\int_D dx\ \ |\phi(x)|\,\mathbb{E}_x\big[\,|\psi(B_{\tau_2}) - \psi(B_t)|\,\mathbf{1}_{\{\tau_2<t\}}\big] \\
&\qquad \le \frac{K'}{t}\int_D dx\ d(x)\mathbb{E}_x\Big[\mathbf{1}_{\{\tau_2<t\}}\mathbb{E}_{B_{\tau_2}}\Big[\sup_{0\le s\le t}|\psi(B_s) - \psi(B_0)|\Big]\Big] \\
&\qquad \le c \sup_{x\in\partial D} \mathbb{E}_x\Big[\sup_{0\le s\le t}|\psi(B_s) - \psi(x)|\Big].
\end{aligned}$$

Let $\varepsilon > 0$. As $\psi \in \mathcal{C}(\bar{D})$ and $\bar{D}$ is compact, $\psi$ is uniformly continuous on $\bar{D}$ and hence there exists $\delta > 0$, such that, $|\psi(y) - \psi(x)| < \varepsilon$ for all $x, y \in \bar{D}$ with $|x - y| < \delta$. Then, we have

$$\begin{aligned}
\sup_{x\in\partial D} \mathbb{E}_x\Big[\sup_{0\le s\le t}|\psi(B_s) - \psi(x)|\Big] &\le \varepsilon + \sup_{x\in\partial D} \mathbb{E}_x\Big[\sup_{0\le s\le t}|\psi(B_s) - \psi(x)|\,\mathbf{1}_{\{\sup_{0\le s\le t}|B_s - x|>\delta\}}\Big] \\
&\le \varepsilon + 2\,\|\,\psi\,\|_\infty \sup_{x\in\partial D} \mathbb{P}_x\Big(\sup_{0\le s\le t}|B_s - x| > \delta\Big).
\end{aligned}$$

And therefore it follows by (3.32), that

$$\lim_{t\to 0} \sup_{x\in\partial D} \mathbb{E}_x\Big[\sup_{0\le s\le t}|\psi(B_s) - \psi(x)|\Big] = 0.$$

This completes the proof. □

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# Index of notation

**Sets and sigma-fields.** Let $E$ be a Polish space.

| | |
|---|---|
| $\mathbb{R}$ | real numbers |
| $\mathbb{R}_+$ | non-negative real numbers $[0, \infty)$ including 0 |
| $\mathcal{B}(E)$ | Borel-sigma-field on $E$ |

**Sets of functions.** The subscript $+$ added to any of these spaces, indicates the subset of *non-negative* functions.

| | |
|---|---|
| $\mathcal{B}(E)$ | Borel-measurbale functions defined on $E$ |
| $\mathcal{B}^b(E)$ | bounded Borel-measurbale functions defined on $E$ |
| $\mathcal{C}(E)$ | continuous functions |
| $\mathcal{C}^b(E)$ | bounded continuous functions |
| $\mathcal{C}^k(E)$ | $k$-times continuously differentiable functions |
| $\mathcal{C}^\infty(E)$ | infinitely differentiable functions |
| $\mathcal{D}(\mathbb{R}_+, E)$ | cádlàg functions $\omega : \mathbb{R}_+ \to E$ |
| $H_b$ | $\psi \in \mathcal{B}^b_+(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\operatorname{supp} \psi \subseteq [0, T] \times \mathbb{R}^d$ for some $T > 0$ |

**Measures.**

| | |
|---|---|
| $\delta_x$ | Dirac mass at the point $x$ |
| $\mathcal{M}(E)$ | Borel measures on $E$ |
| $\mathcal{M}_f(E)$ | finite Borel measures on $E$ |
| $\mathcal{M}^p(E)$ | purely atomic measures on $E$ |
| $\mathcal{M}^p_f(E)$ | purely atomic and finite measures on $E$ |

**Catalytic super-Brownian motion in $\mathbb{R}^d$.**

| | |
|---|---|
| $W$ | Brownian motion in $\mathbb{R}^d$ |
| $\sigma, A$ | catalytic measure and associated branching functional |
| $\mathcal{S}$ | the support of $A$ which is equal to the *fine* support of $\sigma$ |
| $\tau = \tau_{\mathcal{S}}$ | first hitting time of $\mathcal{S}$ by $W$ |
| $\mu, L$ | capacitary measure of the set $\mathcal{S}$ and associated additive functional |
| $X, Z$ | $\sigma$-catalytic super-Brownian motion |
| $\mathcal{L}_{\rho,X}$ | collision local time of $X$ with a uniformly non-polar measure $\rho$ |
| $U$ | non-catalytic superprocess with spatial motion $\xi_t = (A_t^{-1}, W \circ A_t^{-1})$ |
| $\Gamma_\sigma$ | $\int_0^\infty ds\, U_s$, total occupation measure of $U$ |

**Catalytic super-Brownian motion in $D$ and boundary value problems.**

| | |
|---|---|
| $D$ | bounded domain in $\mathbb{R}^d, d \geq 2$ with $\mathcal{C}^3$-boundary |
| $\mathcal{C}^p(D)$ | set of $p$-times continuously differentiable functions defined on $D$ |
| $\mathcal{C}^p(\bar{D})$ | functions of $\mathcal{C}^p(D)$ such that their partial derivatives if $i$-th order ($i \leq p$) extend continuously to $\bar{D}$ |
| $B$ | reflecting Brownian motion in a bounded domain $D \subseteq \mathbb{R}^d$ |
| $\ell$ | local time on the boundary $\partial D$ of $D$ |
| $\sigma$ | surface measure on $\partial D$; the Revuz measure of $\ell$ |
| $F_1, F_2$ | nontrivial partition of $\partial D$ |
| $\ell^1, \ell^2$ | local times on $F_1$ and $F_2$ respectively |
| $\mu, L$ | capacitary measure of the set $F_1$ and the associated additive functional |
| $Z$ | $F_1$-catalytic super-Brownian motion |
| $\ell^*$ | local time on $F_1$ of $B$ killed on $F_2$ |
| $Z^{\mathrm{Dir}}$ | the exit measure on $F_2$ |
| $Z^{\mathrm{Neu}}$ | the Neumann boundary measure on $F_2$ |
| $Z^{\mathrm{Neu}}_\theta$ | the Neumann boundary measure of the approximating problem |
| $S_1, S_2$ | test functions to define 'weak solution' |

**Some special functions.**

| | |
|---|---|
| $\mathbf{1}_A$ | indicator function of the set $A$ |
| $p_t(x, y)$ | heat kernel in $\mathbb{R}^d$ |
| $\bar{p}_t(x, y)$ | transition density of reflecting Brownian motion in a bounded domain |
| $H^x$ | excursion measure of excursions staring in $x$ |